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Nonlinear oscillations and chaotic dynamics of an antisymmetric cross-ply laminated composite rectangular thin plate under parametric excitation

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Abstract

In this paper, nonlinear oscillations and chaotic dynamics of a simply supported antisymmetric cross-ply laminated composite rectangular thin plate under parametric excitation is investigated for the first time. The governing equations of motion for the antisymmetric cross-ply laminated composite plate are derived by using von Karman-type plate equation. The geometric nonlinearity and nonlinear damping are included in the governing equations of motion. A two dof parametrically excited nonlinear system including the quadratic and cubic nonlinear terms is obtained by using the Galerkin method. Based on the Fourier expansion and the temporal rescaling, an asymptotic perturbation method is utilized to obtain four-dimensional nonlinear averaged equations on the amplitude and the phase of nonlinear oscillations of the antisymmetric cross-ply laminated composite plate. Based on the averaged equations, the steady-state nonlinear responses and their stabilities are determined by using numerical approach. The relations between the steady-state nonlinear responses and the amplitude and frequency of parametric excitation are obtained. Under the certain conditions, the antisymmetric cross-ply laminated composite plate may have two steady-state nonzero solutions in which the jumping phenomenon occurs. Numerical simulation is used to discover the periodic and chaotic motions in the antisymmetric cross-ply laminated composite

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rectangular thin plate. It is observed from the numerical results that the multipulse orbits exist in the antisymmetric cross-ply laminated composite rectangular thin plate.

1. Introduction

Due to laminated composite plates having the light weight and high stiffness, they have been widely used in many engineering structures, such as large space station, aircraft, automotive, robot arm and underwater structure. These engineering applications have motivated the interest of researchers in introducing new mathematical models and investigating approaches to predict the complex nonlinear dynamics of laminated composite plates in the case of large deformation. The fiber orientation, the number of layers and their lay-up in laminated composite thin plates introduce many complicated nonlinear dynamic behavior, for example, twisting–bending coupling and stretching–bending coupling of materials. In order to use correctly and effectively laminated composite thin plates in engineering applications, it is necessary to consider different kinds of laminated composite thin plates, such as symmetric and antisymmetric cross-ply laminated composite thin plates. Because the governing differential equations of transverse motion for the antisymmetric cross-ply laminated composite rectangular thin plate are different from ones for symmetric cross-ply laminated composite rectangular thin plate, their nonlinear dynamic characteristics are also different.

In the past two decades, there have been several researches on nonlinear dynamics of the single or multiple modes of laminated composite plates. In [1,2], we studied nonlinear dynamics of a parametrically excited simply supported symmetric cross-ply laminated composite rectangular thin plate. Bhimaraddi [3] utilized the van Karman's plate theory to establish the governing equations of motion for imperfect antisymmetric angle-ply laminated composite plates and analyzed the effect of initial imperfections on the responses of laminated composite plates. Shih and Blotter [4] gave nonlinear vibration analysis of arbitrarily laminated rectangular thin plates on elastic foundations with simply supported and clamped boundary conditions. Flutter of geometrically imperfect shear deformable laminated flat panels was investigated by Chandiramani et al. [5] using the Galerkin method and arclength continuation method. Oh and Nayfeh [6] used experimental method to study nonlinear combination resonances in cantilever laminated composite plates to a harmonic transverse excitation. Abe et al. [7] utilized the method of multiple scales to investigate the two-mode response of simply supported rectangular thin laminated plates subjected to harmonic excitation. Ribeiro and Petyt [8] employed the hierarchical finite element approach to investigate multi-mode geometric nonlinear free oscillations of fully clamped laminated composite plates. In Refs. [9,10], Ganapathi et al. used the finite element method to study the nonlinear instability behavior of laminated composite plate subjected to periodic in-plane load. Recently, Harras and Benamar [11] investigated geometrically nonlinear free oscillations of fully clamped symmetrically laminated rectangular composite plates and analyzed the effect of nonlinearity on the resonant frequencies and the nonlinear fundamental mode shape. Tanriover and Senocak [12] gave large deflection analysis of unsymmetrically laminated composite plate by using the Galerkin method and Newton–Raphson method.

At the same time, researchers have done many investigations on nonlinear oscillations, bifurcations, and the chaos of the thin plate and shallow arch. Hadian and Nayfeh [13] studied asymmetric response of nonlinear clamped circular plates subjected to harmonic excitations and considered the case of a combination-type internal resonance. Feng and Sethna [14] made use of a global perturbation method to study the global bifurcations and chaotic dynamics of thin plate under parametric excitation and obtained the conditions in which the Silnikov-type homoclinic orbits and chaos can occur. Chang et al. [15] investigated the bifurcations and chaos of a rectangular thin plate with 1:1 internal resonance. Sassi and Ostiguy [16] studied the effects of initial geometric imperfections on the interaction between forced and parametric nonlinear oscillations for a simply supported rectangular thin plate subjected to in-plane periodic excitation. Malhotra and Sri Namachchivaya [17] used the averaging method and the Melnikov approach to study the local, global bifurcations and chaos of a two dof shallow arch subjected to simple harmonic excitation for the case of internal resonance. Popov et al. [18] investigated the interaction between different modes of shell oscillations and bifurcations under parametric excitation. Anlas and Elbeyli [19] studied nonlinear dynamics of a simply supported rectangular plate subjected to transverse harmonic excitation. Zhang [20] investigated the global bifurcations and chaotic dynamics for a parametrically excited simply supported thin plate.

In the history on the study of nonlinear oscillations, many asymptotic perturbation techniques, such as the averaging method, the method of multiple scales and the harmonic balance method, have been widely used to construct the approximate solutions for weakly nonlinear systems. In general situations, analysis is carried out only up to the first-order approximation since higher-order terms do not have the influence on the qualitative behavior of the asymptotic solutions. However, the quadratic nonlinearities cannot appear in the first-order approximate solutions when they exist in the original nonlinear systems. Therefore, in order to acquire better qualitative and quantitative characteristics of nonlinear systems including the quadratic and cubic nonlinearities, the second-order averaging or perturbation procedure should be considered in the aforementioned situations.

The averaging method [21] has played a very important role in the study of nonlinear oscillations for a long time. The method of multiple scales was systematically introduced by Nayfeh [22] and has been applied extensively to study nonlinear oscillations and bifurcations in the cases of primary, sub-harmonic and super-harmonic resonances. It has been proved that this method is a powerful tool in determining periodic solutions of small amplitude. The harmonic balance method [23] was also used to study the stable and unstable periodic solutions of nonlinear systems. A satisfying expression for the periodic solution is obtained by using a sufficiently large number of harmonic terms, which often leads to complicated algebraic manipulations. However, the computer symbolic programs can be applied to construct the resulting nonlinear algebraic equations. In a series of papers [24–28], an asymptotic perturbation method was developed by Maccari based on the slow temporal rescalings and balancing of harmonic terms with a simple iteration. In the certain sense, the asymptotic perturbation method can be considered as an attempt to link the most useful characteristics of the harmonic balance method and the method of multiple scales. The asymptotic perturbation method provides a systematic means to obtain increasingly accurate solution by increasing the order of approximation in terms of the small parameter ε. This processes is finished by three steps: (1) obtaining the form of solution in term of harmonic components; (2) introducing a slow time-scale; and (3) solving directly for the various harmonic components via harmonic balance.

The objective of this paper is to consider nonlinear oscillations and chaotic dynamics of a simply supported at the four-edges, antisymmetric cross-ply laminated composite rectangular thin plate subjected to parametric excitation. Because the quadratic and cubic nonlinear terms exist in the governing differential equations of transverse motion, different studying method from one used for symmetric cross-ply laminated composite rectangular thin plate will be utilized to find new nonlinear phenomena in this paper. The study is focused on the case of 1:1 internal resonance and primary parametric resonance. First, considering geometric nonlinearity and nonlinear damping, the governing partial differential equations of motion for the antisymmetric cross-ply laminated composite rectangular thin plate are derived based on the von Karman-type nonlinear plate equation. A parametrically excited two dof nonlinear system including the quadratic and cubic nonlinear terms is obtained by using the Galerkin method. Then, the asymptotic perturbation method developed by Maccari [24–28] is used to reduce the second-order nonautonomous nonlinear differential equations including the quadratic and cubic nonlinear terms to the first-order averaged equations. The stability of steady-state solutions is analyzed. Finally, using numerical method, the averaged equations are analyzed to obtain the steady-state nonlinear responses for the antisymmetric cross-ply laminated composite rectangular thin plate. The results obtained here reveal that under the certain conditions laminated composite rectangular thin plate may have two steady-state nonzero solutions in which the jumping phenomenon occurs. Numerical simulation is also used to discover the periodic and chaotic motions in the antisymmetric cross-ply laminated composite rectangular thin plate. It is observed from the numerical results that the multipulse orbits exist in the antisymmetric cross-ply laminated composite rectangular thin plate.

2. Formulation for equations of motion

A simply supported at the four-edges antisymmetric cross-ply laminated composite rectangular thin plate subjected to in-plane excitation is considered, as shown in Fig. 1. The laminated composite plate under considered here consists of n viscoelastic layers and is thought to be macroscopically homogeneous and orthotropic, where n is an even number. In order to simplify

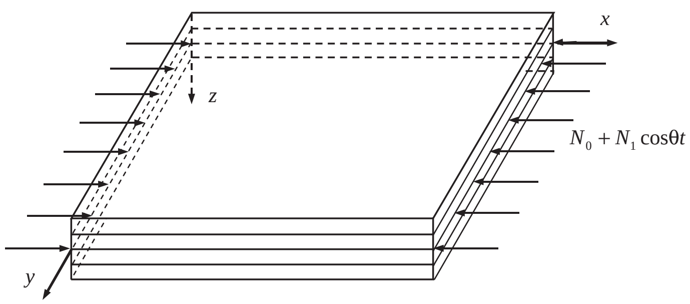

Fig. 1. The model of a rectangular antisymmetric cross-ply laminated composite plate.

the following analytical procedure, it is thought that the thickness of each layer is same. The edge width and length of the plate in the x and y directions are, respectively, a and b and the thickness is h. A Cartesian coordinate Oxy is located in the middle surface of the plate. Assume that the u, v, and w represent the displacements of a point in the middle surface of the plate in the x, y, and z directions, respectively. The excitation in-plane of the plate is distributed along the y direction at $x = 0$ and a and is of the form $-(N_0 + N_1 \cos \theta t)$. Let T_x represent the nonlinear damping which is distributed along the y direction at $x = 0$ and a.

Classical plate theory is used extensively to analyze the plates whose length-to-thickness ratio is of the order of 25 or greater, that is, the thin plates [29]. Because the main interest in this paper is focused in the analytical and numerical researches on nonlinear oscillations and chaotic dynamics of the four-edges antisymmetric cross-ply laminated composite rectangular thin plate subjected to in-plane excitation, we will not consider shear deformation in order to simply the following analysis. When the antisymmetric cross-ply laminated composite rectangular thin plate is simultaneously subjected to in-plane and transverse excitations, it is necessary to consider shear deformation of the antisymmetric cross-ply laminated composite rectangular thin plate. In addition, the displacement field approximations across the laminate thickness is utilized in the following investigation.

According to the von Karman-type plate theory and assumption on flexible plate [30–32], the nonlinear equations of motion for the antisymmetric cross-ply laminated composite rectangular thin plate in-plane and out-of-plane are given as

$$N_{x,x} + N_{xy,y} = 0, \tag{1a}$$

$$N_{xy,x} + N_{y,y} = 0, \tag{1b}$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} - \zeta w_{,t} - \rho h w_{,tt} = 0, \tag{1c}$$

where N_x, N_y and N_{xy} represent membrane forces, M_x, M_y and M_{xy} represent flexural moments, ρ is the density of the plate and ζ is the linear damping coefficient.

For the case of regular, antisymmetric cross-ply laminated composite plate with viscoelastic characteristic, the force and moment resultants, which are related to the strains and curvatures, are expressed as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & -B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & -B_{11} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix}. \tag{2}$$

It is noticed from Eq. (2) that the expression of the force and moment resultants for antisymmetric cross-ply laminated composite plate is different from one for symmetric cross-ply laminated composite plate.

It is known that the extensional stiffnesses A_{ij} ($i = 1$, 2 and $j = 1$,2; $i = 6$ and $j = 6$), the flexural stiffnesses D_{ij} ($i = 1$, 2 and $j = 1$, 2; $i = 6$ and $j = 6$), and the flexural–extensional coupling stiffnesses B_{ij} $(i=1,\ j=1;\ i=2,\ j=2)$ are, respectively, denoted by

$$A_{ij}=\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}(Q_{ij})_k\,\mathrm{d}z,\quad (i=1,2,\ j=1,2;\ i=6,\ j=6), \tag{3}$$

$$D_{ij}=\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}z^2(Q_{ij})_k\,\mathrm{d}z,\quad (i=1,2,\ j=1,2;\ i=6,\ j=6), \tag{4}$$

$$B_{ij}=\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}z(Q_{ij})_k\,\mathrm{d}z,\quad (i=1,\ j=1;\ i=2,\ j=2), \tag{5}$$

where n is the number of the viscoelastic layers.

The aforementioned coefficients Q_{ij} $(i=1,\ 2$ and $j=1,\ 2;\ i=6$ and $j=6)$ are defined as follows:

$$Q_{11}=\frac{E_1}{1-v_{12}v_{21}},\ Q_{12}=Q_{21}=\frac{v_{12}E_2}{1-v_{12}v_{21}}=\frac{v_{21}E_1}{1-v_{12}v_{21}},\ Q_{22}=\frac{E_2}{1-v_{12}v_{21}},\ Q_{66}=G_{12}, \tag{6}$$

where $E_i (i=1,2)$ are Young's modulus, v_{ij} $(i,j=1,2)$ are Poisson's ratio and G_{12} is shear modulus.

Since there are the membrane forces, the dissipation force exists along the boundaries of laminated composite plate. The dissipation function is assumed to have the following form [33]:

$$D=\int_0^b k\int_0^q f(y,q)\,\mathrm{d}q\,\mathrm{d}y, \tag{7}$$

where k is friction coefficient, q is velocity which can be expressed as

$$q=\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left[\frac{1}{2}\int_0^a\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right], \tag{8}$$

and $f(y,q)$ is assumed to be directly proportional to q. Suppose that dissipation force is uniformly distributed along the y direction at $x=0$ and a. Therefore, the dissipation function can be rewritten as

$$D=\frac{1}{2}\int_0^b k\left\{\frac{\partial}{\partial t}\left[\frac{1}{2}\int_0^a\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]\right\}^2\mathrm{d}y. \tag{9}$$

The nonlinear damping is defined by

$$T_x=-\frac{\partial D}{\partial q}. \tag{10}$$

The boundary conditions of simply supported antisymmetric cross-ply laminated composite rectangular thin plate are expressed as follows:

$$\text{at } x=0,\quad w=u=M_x=N_{xy}=0, \tag{11}$$

$$\text{at } x=a,\ w=M_x=N_{xy}=0 \text{ and } N_x=-\frac{\partial T_x}{\partial y}-(N_0+N_1\cos\theta t), \tag{12}$$

$$\text{at } y=0,\ y=b,\ w=v=M_y=N_{xy}=0. \tag{13}$$

According to the von Karman-type plate theory and assumption on flexible plate [30–32], the strains ε_i $(i=x,y,xy)$ and curvatures k_i $(i=x,y,xy)$ of the middle plane are related to the displacements by

$$\varepsilon_x=u_{,x}+\tfrac{1}{2}(w_{,x})^2,\quad \varepsilon_y=v_{,y}+\tfrac{1}{2}(w_{,y})^2,\quad \varepsilon_{xy}=u_{,y}+v_{,x}+w_{,x}w_{,y}, \tag{14}$$

$$k_x=-w_{,xx},\quad k_{xy}=-2w_{,xy},\quad k_y=-w_{,yy}. \tag{15}$$

Substituting, respectively, Eqs. (2), (14) and (15) into Eq. (1) yields the governing equations of motion for the antisymmetric cross-ply laminated composite rectangular thin plate in-plane and out-of-plane as

$$A_{11}u_{,xx}+A_{66}u_{,yy}+(A_{12}+A_{66})v_{,xy}-B_{11}w_{,xxx}+w_{,x}\left(A_{11}w_{,xx}+A_{66}w_{,yy}\right)+(A_{12}+A_{66})w_{,y}w_{,xy}=0, \tag{16}$$

$$(A_{12}+A_{66})u_{,xy}+A_{11}v_{,yy}+A_{66}v_{,xx}+B_{11}w_{,yyy}+w_{,y}\left(A_{11}w_{,yy}+A_{66}w_{,xx}\right)+(A_{12}+A_{66})w_{,x}w_{,xy}=0, \tag{17}$$

$$\begin{aligned}&\rho h w_{,tt}+\zeta w_{,t}+D_{11}\left(w_{,xxxx}+w_{,yyyy}\right)+2(D_{12}+D_{66})w_{,xxyy}+B_{11}\left(v_{,yyy}-u_{,xxx}\right)\\&\quad+B_{11}\left\lfloor\left(w_{,yy}\right)^2-(w_{,xx})^2+w_{,y}w_{,yyy}-w_{,x}w_{,xxx}\right]-N_xw_{,xx}-N_yw_{,yy}-2N_{xy}w_{,xy}=0.\end{aligned} \tag{18}$$

We mainly consider nonlinear dynamics of the antisymmetric cross-ply laminated composite rectangular thin plate in the first two modes. The experimental results given in [32] indicate that the shape of modes for laminated composite plates is different from one for isotropic material plates due to the effects of the fiber orientation, the number of layers and their lay-up in laminated composite thin plates. In Ref. [30], Chia gave an expression of the first-order mode in the form of square trigonometric terms for the solution of nonlinear oscillations of laminated composite thin plates. It is our desirable to choose a suitable mode function to satisfy the first two modes of nonlinear oscillations and the boundary condition for laminated composite thin plates. Therefore, in our study, we consider the first two modes of antisymmetric cross-ply laminated composite anisotropic plate and express w in form

$$w(x,y,t)=f_1(t)\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi y}{b}\right)+f_2(t)\sin^2\left(\frac{2\pi x}{a}\right)\sin^2\left(\frac{\pi y}{b}\right), \tag{19}$$

where $f_i(t)$ $(i=1,2)$ are, respectively, the amplitudes of the first- and second-order modes.

Substituting Eq. (19) into Eqs. (16) and (17) and considering the boundary condition

$$\int_0^b N_x|_{x=a}\,\mathrm{d}y=\int_0^b\left[-\frac{\partial T_x}{\partial y}-(N_0+N_1\cos\theta t)\right]\mathrm{d}y, \tag{20}$$

we can obtain the expressions of the displacements u and v. Substituting the u, v and w into Eqs. (2), (13) and (14), we obtain the expressions of the N_x, N_y, N_{xy}, M_x, M_y and M_{xy}.

Substituting the aforementioned u, v, N_x, N_y, N_{xy}, M_x, M_y and M_{xy} into Eq. (18) and applying the Galerkin procedure yield the governing differential equations of transverse motion for the antisymmetric cross-ply laminated composite rectangular thin plate

$$\begin{aligned}\frac{3}{2}\ddot{f}_1 + \ddot{f}_2 + \frac{3}{2}\eta(1 + \mu_{11}f_1^2 + \mu_{12}f_1f_2)\dot{f}_1 + \eta(1 + \mu_{13}f_1f_2 + \mu_{14}f_2^2)\dot{f}_2 \\ + \Omega_{11}^2(1 - 2h_{11}\cos\theta t)f_1 + \Omega_{12}^2(1 - 2h_{12}\cos\theta t)f_2 + \varphi_{11}f_1^2 + \varphi_{12}f_1f_2 \\ + \varphi_{13}f_2^2 + \varphi_{14}f_1^3 + \varphi_{15}f_1^2f_2 + \varphi_{16}f_1f_2^2 + \varphi_{17}f_2^3 = 0,\end{aligned} \tag{21a}$$

$$\begin{aligned}\ddot{f}_1 + \frac{3}{2}\ddot{f}_2 + \eta(1 + \mu_{21}f_1^2 + \mu_{22}f_1f_2)\dot{f}_1 + \frac{3}{2}\eta(1 + \mu_{23}f_1f_2 + \mu_{24}f_2^2)\dot{f}_2 \\ + \Omega_{21}^2(1 - 2h_{21}\cos\theta t)f_1 + \Omega_{22}^2(1 - 2h_{22}\cos\theta t)f_2 + \varphi_{21}f_1^2 + \varphi_{22}f_1f_2 \\ + \varphi_{23}f_2^2 + \varphi_{24}f_1^3 + \varphi_{25}f_1^2f_2 + \varphi_{26}f_1f_2^2 + \varphi_{27}f_2^3 = 0,\end{aligned} \tag{21b}$$

where $\eta = \zeta/\rho h$, the coefficients μ_{ij} $(i = 1,2;\ j = 1,\ldots,4)$, h_{ij} $(i,j = 1,2)$, Ω_{ij} $(i,j = 1,2)$, φ_{1i} $(i = 1,\ldots,7)$ and φ_{2i} $(i = 1,\ldots,7)$ can be found in Appendix A.

In order to obtain the dimensionless equations, we introduce the transformations of variables and parameters

$$\tilde{t} = \Omega_{11}t, \quad x_i = \frac{(ab)^{1/2}}{h^2}f_i, \quad \Omega = \frac{\theta}{\Omega_{11}}. \tag{22}$$

For simplicity, we drop the tilde in the following analysis. Substituting Eq. (22) into Eq. (21), we obtain the equations of transverse motion of the antisymmetric cross-ply laminated composite rectangular thin plate for the dimensionless as follows:

$$\begin{aligned}\ddot{x}_1 + (\alpha_{11} + \alpha_{12}x_1^2 + \alpha_{13}x_1x_2)\dot{x}_1 + (\alpha_{14}x_1x_2 + \alpha_{15}x_2^2)\dot{x}_2 + (\beta_{11} - \beta_{12}\cos\Omega\tau)x_1 \\ + (\beta_{13} - \beta_{14}\cos\Omega\tau)x_2 + \gamma_{11}x_1^3 + \gamma_{12}x_1^2x_2 + \gamma_{13}x_1x_2^2 + \gamma_{14}x_2^3 + \gamma_{15}x_1^2 + \gamma_{16}x_1x_2 + \gamma_{17}x_2^2 = 0,\end{aligned} \tag{23a}$$

$$\begin{aligned}\ddot{x}_2 + (\alpha_{21}x_1^2 + \alpha_{22}x_1x_2)\dot{x}_1 + (\alpha_{23} + \alpha_{24}x_1x_2 + \alpha_{25}x_2^2)\dot{x}_2 + (\beta_{21} - \beta_{22}\cos\Omega\tau)x_1 \\ + (\beta_{23} - \beta_{24}\cos\Omega\tau)x_2 + \gamma_{21}x_1^3 + \gamma_{22}x_1^2x_2 + \gamma_{23}x_1x_2^2 + \gamma_{24}x_2^3 + \gamma_{25}x_1^2 + \gamma_{26}x_1x_2 + \gamma_{27}x_2^2 = 0,\end{aligned} \tag{23b}$$

where α_{ij} $(i = 1,2;\ j = 1,\ldots,5)$, β_{ij} $(i = 1,2;\ j = 1,\ldots,4)$ and γ_{ij} $(i = 1,2;\ j = 1,\ldots,7)$ are dimensionless coefficients, which are presented by the Appendix B.

The above equations, which include the quadratic, cubic terms and parametric excitations, describe nonlinear vibrations of the antisymmetric cross-ply laminated composite rectangular thin plate in the first two modes.

Comparing the governing differential equations of transverse motion between the antisymmetric and symmetric cross-ply laminated composite rectangular thin plates, it is observed that the quadratic terms exist for the antisymmetric cross-ply laminated composite rectangular thin plate. In the case of symmetric cross-ply laminated composite rectangular thin plate, the quadratic terms do not exist in the governing differential equations of transverse motion. In order to consider the influence of the quadratic terms on nonlinear dynamic characteristics of the antisymmetric cross-ply laminated composite rectangular thin plate, we need to obtain the second-order approximate solution of Eq. (23). Up to now, it is difficult to utilize the method of multiple scales to obtain the second-order approximate solution of Eq. (23). Therefore, in the following analysis, we will utilize the asymptotic perturbation method presented by Maccari [24–28] to look for the second-order approximate solution of Eq. (23). Using the asymptotic perturbation method, we can obtain increasingly accurate solution by increasing the order of approximation in terms of the small parameter ε.

3. Asymptotic perturbation method and second-order approximate solution

In the following analysis, we use the asymptotic perturbation method [24–28] to obtain the averaged equations of Eq. (23). Based on the averaged equations, numerical approach is utilized to find the periodic motions and local bifurcations in the antisymmetric cross-ply laminated composite rectangular thin plate for the first two modes.

Considering the case of primary parameter resonance and 1:1 internal resonance, we have the following relations:

$$\omega_1 = \frac{\Omega}{2} + \varepsilon^2\sigma_1, \quad \omega_2 = \frac{\Omega}{2} + \varepsilon^2\sigma_2, \quad \omega_1 \approx \omega_2 \tag{24}$$

where $\omega_1^2 = \beta_{11}$ and $\omega_2^2 = \beta_{23}$ are two linear natural frequencies, σ_1 and σ_2 are two detuning parameters.

In order to utilize the asymptotic perturbation method to analyze nonlinear responses of Eq. (23), the scale transformations may be introduced as

$$\begin{aligned} &\alpha_{11} \to \varepsilon^2\alpha_{11}, \quad \alpha_{23} \to \varepsilon^2\alpha_{23}, \quad \beta_{12} \to \varepsilon^2\beta_{12}, \quad \beta_{13} \to \varepsilon^2\beta_{13}, \\ &\beta_{14} \to \varepsilon^2\beta_{14}, \quad \beta_{21} \to \varepsilon^2\beta_{21}, \quad \beta_{22} \to \varepsilon^2\beta_{22}, \quad \beta_{24} \to \varepsilon^2\beta_{24}. \end{aligned} \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (23) yields

$$\begin{aligned} &\ddot{x} + (\varepsilon^2\alpha_{11} + \alpha_{12}x^2 + \alpha_{13}xy)\dot{x} + (\alpha_{14}xy + \alpha_{15}y^2)\dot{y} + [(\omega + \varepsilon^2\sigma_1)^2 - \varepsilon^2\beta_{12}\cos\Omega t]x \\ &\quad + \varepsilon^2(\beta_{13} - \beta_{14}\cos\Omega t)y + \gamma_{11}x^3 + \gamma_{12}x^2y + \gamma_{13}xy^2 + \gamma_{14}y^3 + \gamma_{15}x^2 + \gamma_{16}xy + \gamma_{17}y^2 = 0, \end{aligned} \tag{26a}$$

$$\begin{aligned} &\ddot{y} + (\alpha_{21}x^2 + \alpha_{22}xy)\dot{x} + (\varepsilon^2\alpha_{23} + \alpha_{24}xy + \alpha_{25}y^2)\dot{y} + \varepsilon^2(\beta_{21} - \beta_{22}\cos\Omega t)x + (\omega + \varepsilon^2\sigma_2)^2 y \\ &\quad - \varepsilon^2\beta_{24}y\cos\Omega t + \gamma_{21}x^3 + \gamma_{22}x^2y + \gamma_{23}xy^2 + \gamma_{24}y^3 + \gamma_{25}x^2 + \gamma_{26}xy + \gamma_{27}y^2 = 0, \end{aligned} \tag{26b}$$

where $\omega = \Omega/2$.

Now we introduce the temporal rescaling

$$\tau = \varepsilon^q t, \tag{27}$$

where q is a rational positive number, which will be fixed afterwards. The value of q fixes the magnitude order of the temporal asymptotic limit in such a way that the nonlinear effects become consistent and not negligible. If $t \to \infty$, we set $\varepsilon \to 0$, so the value of τ keeps finite.

Taking $\varepsilon = 0$ in Eq. (26) and neglecting all nonlinear terms, it is found that resulting linear equation has simple harmonic solutions

$$x(t) = A_1\exp(-\mathrm{i}\omega t) + \mathrm{cc}, \quad y(t) = A_2\exp(-\mathrm{i}\omega t) + \mathrm{cc}, \tag{28}$$

where A_1 and A_2 are two constants depending on initial conditions and cc represents the parts of complex conjugate. It is considered that the nonlinear effects will induce the modulation of the amplitudes A_1 and A_2 and the appearance of higher harmonics. The slow modulation resulted from the nonlinear terms can be determined by means of the rescaled time (27).

Assuming that solutions $x(t)$ and $y(t)$ of Eq. (26) can be expressed in a power series of small parameter ε, we have

$$x(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{r_n} \psi_n(\tau, \varepsilon) \exp(-\mathrm{i}n\omega t), \tag{29a}$$

$$y(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{r_n} \phi_n(\tau, \varepsilon) \exp(-\mathrm{i}n\omega t), \tag{29b}$$

where $r_n = |n|$ for $n \neq 0$, and $r_0 = r$ is a positive number which will be fixed later on.

Considering the real part of solutions $x(t)$ and $y(t)$ yields

$$\psi_n(\tau, \varepsilon) = \psi^*_{-n}(\tau, \varepsilon), \tag{30a}$$

$$\phi_n(\tau, \varepsilon) = \phi^*_{-n}(\tau, \varepsilon), \tag{30b}$$

where the asterisk denotes complex conjugate.

Therefore, the assumed solution (29) can be rewritten more explicitly as follows

$$\begin{aligned} x(t) = {} & \varepsilon^r \psi_0(\tau, \varepsilon) + (\varepsilon \psi_1(\tau, \varepsilon) \exp(-\mathrm{i}\omega t) + \varepsilon^2 \psi_2(\tau, \varepsilon) \exp(-2\mathrm{i}\omega t) \\ & + \varepsilon^3 \psi_3(\tau, \varepsilon) \exp(-3\mathrm{i}\omega t) + \varepsilon^4 \psi_4(\tau, \varepsilon) \exp(-4\mathrm{i}\omega t) + \mathrm{cc}) + O(\varepsilon^5), \end{aligned} \tag{31a}$$

$$\begin{aligned} y(t) = {} & \varepsilon^r \phi_0(\tau, \varepsilon) + (\varepsilon \phi_1(\tau, \varepsilon) \exp(-\mathrm{i}\omega t) + \varepsilon^2 \phi_2(\tau, \varepsilon) \exp(-2\mathrm{i}\omega t) \\ & + \varepsilon^3 \phi_3(\tau, \varepsilon) \exp(-3\mathrm{i}\omega t) + \varepsilon^4 \phi_4(\tau, \varepsilon) \exp(-4\mathrm{i}\omega t) + \mathrm{cc}) + O(\varepsilon^5). \end{aligned} \tag{31b}$$

We find that the solutions have been considered as the combination of different harmonics with the coefficients depending on both τ and ε. Suppose that functions $\psi_n(\tau, \varepsilon)$ and $\phi_n(\tau, \varepsilon)$ can be expanded in power series of small parameter ε

$$\psi_n(\tau, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \psi_n^{(i)}(\tau), \tag{32a}$$

$$\phi_n(\tau, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \phi_n^{(i)}(\tau). \tag{32b}$$

It is also assumed that the limits of functions $\psi_n(\tau, \varepsilon)$ and $\phi_n(\tau, \varepsilon)$ as $\varepsilon \to 0$ exist and are finite. For simplicity of analysis, we use abbreviations $\psi_n^{(0)} = \psi_n$ and $\phi_n^{(0)} = \phi_n$ for $n \neq 1$ and $\psi_1^{(0)} = \psi$, $\phi_1^{(0)} = \phi$ for $n = 1$. Note that the introduction of the temporal rescaling (27) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} (\psi_n \exp(-\mathrm{i}n\omega t)) = \left(-\mathrm{i}n\omega \psi_n + \varepsilon^q \frac{\mathrm{d}\psi_n}{\mathrm{d}\tau} \right) \exp(-\mathrm{i}n\omega t), \tag{33a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_n \exp(-\mathrm{i}n\omega t)) = \left(-\mathrm{i}n\omega\phi_n + \varepsilon^q \frac{\mathrm{d}\phi_n}{\mathrm{d}\tau}\right)\exp(-\mathrm{i}n\omega t). \tag{33b}$$

In order to determine coefficients $\psi_n(\tau, \varepsilon)$ and $\phi_n(\tau, \varepsilon)$, we substitute solution (29) into Eq. (26) and obtain the equations for each harmonic with order n and for a fixed order of approximation on the perturbation parameter ε.

For $n = 0$, we obtain

$$\varepsilon^r\psi_0\omega^2 + 2\varepsilon^2\gamma_{15}|\psi|^2 + \varepsilon^2\gamma_{16}(\psi\phi^* + \psi^*\phi) + 2\varepsilon^2\gamma_{17}|\phi|^2 = 0, \tag{34a}$$

$$\varepsilon^r\phi_0\omega^2 + 2\varepsilon^2\gamma_{25}|\psi|^2 + \varepsilon^2\gamma_{26}(\psi\phi^* + \psi^*\phi) + 2\varepsilon^2\gamma_{27}|\phi|^2 = 0. \tag{34b}$$

The correct balance of terms results $r = 2$ and the following relations are obtained:

$$\psi_0 = -\frac{2\gamma_{15}|\psi|^2 + \gamma_{16}(\psi\phi^* + \psi^*\phi) + 2\gamma_{17}|\phi|^2}{\omega^2}, \tag{35a}$$

$$\phi_0 = -\frac{2\gamma_{25}|\psi|^2 + \gamma_{26}(\psi\phi^* + \psi^*\phi) + 2\gamma_{27}|\phi|^2}{\omega^2}. \tag{35b}$$

For $n = 2$, taking into account Eq. (33) yields

$$-3\varepsilon^2\omega^2\psi_2 + \varepsilon^2\gamma_{15}\psi^2 + \varepsilon^2\gamma_{16}\psi\phi + \varepsilon^2\gamma_{17}\phi^2 = 0, \tag{36a}$$

$$-3\varepsilon^2\omega^2\phi_2 + \varepsilon^2\gamma_{25}\psi^2 + \varepsilon^2\gamma_{26}\psi\phi + \varepsilon^2\gamma_{27}\phi^2 = 0 \tag{36b}$$

and the correspondent relations

$$\psi_2 = \frac{\gamma_{15}\psi^2 + \gamma_{16}\psi\phi + \gamma_{17}\phi^2}{3\omega^2}, \tag{37a}$$

$$\phi_2 = \frac{\gamma_{25}\psi^2 + \gamma_{26}\psi\phi + \gamma_{27}\phi^2}{3\omega^2}. \tag{37b}$$

For $n = 1$, based on Eq. (26), we have

$$\begin{aligned} &-2\varepsilon^{1+q}\mathrm{i}\omega\dot{\psi} - \mathrm{i}\varepsilon^3\alpha_{11}\omega\psi - \mathrm{i}\varepsilon^3\alpha_{12}\omega|\psi|^2\psi - \mathrm{i}\varepsilon^3\alpha_{13}\omega\psi^2\phi^* - \mathrm{i}\varepsilon^3\alpha_{14}\omega\phi^2\psi^* - \mathrm{i}\varepsilon^3\alpha_{15}\omega|\phi|^2\phi \\ &+ 2\varepsilon^3\omega\sigma_1\psi + \varepsilon^3\beta_{13}\phi + 3\varepsilon^3\gamma_{11}|\psi|^2\psi + 2\varepsilon^3\gamma_{12}|\psi|^2\phi + \varepsilon^3\gamma_{12}\psi^2\phi^* + 2\varepsilon^3\gamma_{13}\psi|\phi|^2 + \varepsilon^3\gamma_{13}\phi^2\psi^* \\ &+ 3\varepsilon^3\gamma_{14}|\phi|^2\phi - \varepsilon^3\,\frac{\beta_{14}\phi^* + \beta_{12}\psi^*}{2} + 2\gamma_{15}(\varepsilon^3\psi^*\psi_2 + \varepsilon^{1+r}\psi_0\psi) + \gamma_{16}\varepsilon^3(\psi^*\phi_2 + \psi_2\phi^*) \\ &+ \gamma_{16}\varepsilon^{1+r}(\psi_0\phi + \psi\phi_0) + 2\gamma_{17}\varepsilon^3(\phi_0\phi + \phi^*\phi_2) = 0, \end{aligned} \tag{38a}$$

$$\begin{aligned} &-2\mathrm{i}\varepsilon^{1+q}\omega\dot{\phi} - \mathrm{i}\varepsilon^3\alpha_{23}\omega\phi - \mathrm{i}\varepsilon^3\alpha_{21}\omega|\psi|^2\psi - \mathrm{i}\varepsilon^3\alpha_{22}\omega\psi^2\phi^* - \mathrm{i}\varepsilon^3\alpha_{24}\omega\phi^2\psi^* - \mathrm{i}\varepsilon^3\alpha_{25}\omega|\phi|^2\phi \\ &+ 2\varepsilon^3\omega\sigma_2\phi + \varepsilon^3\beta_{21}\psi + 3\varepsilon^3\gamma_{21}|\psi|^2\psi + 2\varepsilon^3\gamma_{22}|\psi|^2\phi + \varepsilon^3\gamma_{22}\psi^2\phi^* + 2\varepsilon^3\gamma_{23}\psi|\phi|^2 + \varepsilon^3\gamma_{23}\phi^2\psi^* \\ &+ 3\varepsilon^3\gamma_{24}|\phi|^2\phi - \varepsilon^3\,\frac{\beta_{24}\phi^* + \beta_{22}\psi^*}{2} + 2\gamma_{25}(\varepsilon^3\psi^*\psi_2 + \varepsilon^{1+r}\psi_0\psi) + \gamma_{26}\varepsilon^3(\psi^*\phi_2 + \psi_2\phi^*) \\ &+ \gamma_{26}\varepsilon^{1+r}(\psi_0\phi + \psi\phi_0) + 2\gamma_{27}\varepsilon^3(\phi_0\phi + \phi^*\phi_2) = 0. \end{aligned} \tag{38b}$$

We find that in the case $q = 2$, the first terms of Eq. (38) have same magnitude order as all other nonlinear terms.

Based on Eqs. (35) and (37), the differential equation for the evolution of the complex amplitudes ψ and ϕ can be derived as

$$\frac{d\psi}{d\tau} = (a_1 + i\tilde{a}_1)\psi + (b_1 + i\tilde{b}_1)|\psi|^2\psi + i\tilde{c}_1|\psi|^2\phi + i\tilde{d}_1|\phi|^2\psi + (e_1 + i\tilde{e}_1)|\phi|^2\phi + (f_1 + i\tilde{f}_1)\psi^2\phi^* + (g_1 + i\tilde{g}_1)\phi^2\psi^* + i\tilde{h}_1\phi + i\tilde{i}_1\psi^* + i\tilde{j}_1\phi^*, \quad (39a)$$

$$\frac{d\phi}{d\tau} = (a_2 + i\tilde{a}_2)\phi + (b_2 + i\tilde{b}_2)|\psi|^2\psi + i\tilde{c}_2|\psi|^2\phi + i\tilde{d}_2|\phi|^2\psi + (e_2 + i\tilde{e}_2)|\phi|^2\phi + (f_2 + i\tilde{f}_2)\psi^2\phi^* + (g_2 + i\tilde{g}_2)\phi^2\psi^* + i\tilde{h}_2\psi + i\tilde{i}_2\psi^* + i\tilde{j}_2\phi^*, \quad (39b)$$

where the coefficients presented in Eq. (39) can be found in Appendix C.

In order to transform Eq. (39) into the polar coordinate, let

$$\psi(\tau) = \rho(\tau)\exp(i\alpha(\tau)), \quad (40a)$$

$$\phi(\tau) = \chi(\tau)\exp(i\beta(\tau)). \quad (40b)$$

Substituting Eq. (40) into Eq. (39) and separating the real and imaginary parts, we obtain the averaged equations as follows

$$\frac{d\rho}{d\tau} = a_1\rho + b_1\rho^3 + \tilde{c}_1\rho^2\chi\sin(\alpha-\beta) + e_1\chi^3\cos(\alpha-\beta) + \tilde{e}_1\chi^3\sin(\alpha-\beta) + f_1\rho^2\chi\cos(\alpha-\beta) - \tilde{f}_1\rho^2\chi\sin(\alpha-\beta) + g_1\rho\chi^2\cos(2\alpha-2\beta) + \tilde{g}_1\rho\chi^2\sin(2\alpha-2\beta) + \tilde{h}_1\chi\sin(\alpha-\beta) + \tilde{i}_1\rho\sin(2\alpha) + \tilde{j}_1\chi\sin(\alpha+\beta), \quad (41a)$$

$$\rho\frac{d\alpha}{d\tau} = \tilde{a}_1\rho + \tilde{b}_1\rho^3 + \tilde{c}_1\rho^2\chi\cos(\alpha-\beta) + \tilde{d}_1\rho\chi^2 - e_1\chi^3\sin(\alpha-\beta) + \tilde{e}_1\chi^3\cos(\alpha-\beta) + f_1\rho^2\chi\sin(\alpha-\beta) + \tilde{f}_1\rho^2\chi\cos(\alpha-\beta) - g_1\rho\chi^2\sin(2\alpha-2\beta) + \tilde{g}_1\rho\chi^2\cos(2\alpha-2\beta) + \tilde{h}_1\chi\cos(\alpha-\beta) + \tilde{i}_1\rho\cos(2\alpha) + \tilde{j}_1\chi\cos(\alpha+\beta), \quad (41b)$$

$$\frac{d\chi}{d\tau} = a_2\chi + b_2\rho^3\cos(\alpha-\beta) - \tilde{b}_2\rho^3\sin(\alpha-\beta) - \tilde{d}_2\rho\chi^2\sin(\alpha-\beta) + e_2\chi^3 + f_2\rho^2\chi\cos(2\alpha-2\beta) - \tilde{f}_2\rho^2\chi\sin(2\alpha-2\beta) + g_2\rho\chi^2\cos(\alpha-\beta) + \tilde{g}_2\rho\chi^2\sin(\alpha-\beta) - \tilde{h}_2\rho\sin(\alpha-\beta) + \tilde{i}_2\rho\sin(\alpha+\beta) + \tilde{j}_2\chi\sin(2\beta), \quad (41c)$$

$$\chi\frac{d\beta}{d\tau} = \tilde{a}_2\chi + b_2\rho^3\sin(\alpha-\beta) + \tilde{b}_2\rho^3\cos(\alpha-\beta) + \tilde{c}_2\rho^2\chi + \tilde{d}_2\rho\chi^2\cos(\alpha-\beta) + \tilde{e}_2\chi^3 + f_2\rho^2\chi\sin(2\alpha-2\beta) + \tilde{f}_2\rho^2\chi\cos(2\alpha-2\beta) - g_2\rho\chi^2\sin(\alpha-\beta) + \tilde{g}_2\rho\chi^2\cos(\alpha-\beta) + \tilde{h}_2\rho\cos(\alpha-\beta) + \tilde{i}_2\rho\cos(\alpha+\beta) + \tilde{j}_2\chi\cos(2\beta). \quad (41d)$$

The averaged equation (41) in the polar coordinate form can be denoted in the Cartesian form, let

$$\psi = x_1 + \mathrm{i}x_2, \quad \phi = x_3 + \mathrm{i}x_4, \tag{42}$$

where

$$x_1 = \rho \cos \alpha, \quad x_2 = \rho \sin \alpha, \quad x_3 = \chi \cos \beta, \quad x_4 = \chi \sin \beta. \tag{43}$$

Substituting (42) into Eq. (39), the averaged equations in the Cartesian form are obtained as follows:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = {} & a_1 x_1 + (\tilde{i}_1 - \tilde{a}_1)x_2 + (\tilde{j}_1 - \tilde{h}_1)x_4 + b_1 x_1(x_1^2 + x_2^2) - \tilde{b}_1 x_2(x_1^2 + x_2^2) - \tilde{c}_1 x_4(x_1^2 + x_2^2) \\ & - \tilde{d}_1 x_2(x_3^2 + x_4^2) + e_1 x_3(x_3^2 + x_4^2) - \tilde{e}_1 x_4(x_3^2 + x_4^2) + f_1 x_3(x_1^2 - x_2^2) + \tilde{f}_1 x_4(x_1^2 - x_2^2) \\ & + g_1 x_1(x_3^2 - x_4^2) + \tilde{g}_1 x_2(x_3^2 - x_4^2) + 2f_1 x_1 x_2 x_4 - 2\tilde{f}_1 x_1 x_2 x_3 + 2g_1 x_2 x_3 x_4 - 2\tilde{g}_1 x_1 x_3 x_4,\end{aligned} \tag{44a}$$

$$\begin{aligned}\frac{\mathrm{d}x_2}{\mathrm{d}\tau} = {} & (\tilde{i}_1 + \tilde{a}_1)x_1 + a_1 x_2 + (\tilde{h}_1 + \tilde{j}_1)x_3 + b_1 x_2(x_1^2 + x_2^2) + \tilde{b}_1 x_1(x_1^2 + x_2^2) + \tilde{c}_1 x_3(x_1^2 + x_2^2) \\ & + \tilde{d}_1 x_1(x_3^2 + x_4^2) + \tilde{e}_1 x_3(x_3^2 + x_4^2) + e_1 x_4(x_3^2 + x_4^2) - f_1 x_4(x_1^2 - x_2^2) + \tilde{f}_1 x_3(x_1^2 - x_2^2) \\ & - g_1 x_2(x_3^2 - x_4^2) + \tilde{g}_1 x_1(x_3^2 - x_4^2) + 2f_1 x_1 x_2 x_3 + 2\tilde{f}_1 x_1 x_2 x_4 + 2g_1 x_1 x_3 x_4 + 2\tilde{g}_1 x_2 x_3 x_4,\end{aligned} \tag{44b}$$

$$\begin{aligned}\frac{\mathrm{d}x_3}{\mathrm{d}\tau} = {} & (\tilde{i}_2 - \tilde{h}_2)x_2 + a_2 x_3 + (\tilde{j}_2 - \tilde{a}_2)x_4 + b_2 x_1(x_1^2 + x_2^2) - \tilde{b}_2 x_2(x_1^2 + x_2^2) - \tilde{c}_2 x_4(x_1^2 + x_2^2) \\ & - \tilde{d}_2 x_2(x_3^2 + x_4^2) + e_2 x_3(x_3^2 + x_4^2) - \tilde{e}_2 x_4(x_3^2 + x_4^2) + f_2 x_3(x_1^2 - x_2^2) + \tilde{f}_2 x_4(x_1^2 - x_2^2) \\ & + g_2 x_1(x_3^2 - x_4^2) + \tilde{g}_2 x_2(x_3^2 - x_4^2) + 2f_2 x_1 x_2 x_4 - 2\tilde{f}_2 x_1 x_2 x_3 + 2g_2 x_2 x_3 x_4 - 2\tilde{g}_2 x_1 x_3 x_4,\end{aligned} \tag{44c}$$

$$\begin{aligned}\frac{\mathrm{d}x_4}{\mathrm{d}\tau} = {} & (\tilde{h}_2 + \tilde{i}_2)x_1 + a_2 x_4 + (\tilde{a}_2 + \tilde{j}_2)x_3 + b_2 x_2(x_1^2 + x_2^2) + \tilde{b}_2 x_1(x_1^2 + x_2^2) + \tilde{c}_2 x_3(x_1^2 + x_2^2) \\ & + \tilde{d}_2 x_1(x_3^2 + x_4^2) + e_2 x_4(x_3^2 + x_4^2) + \tilde{e}_3 x_4(x_3^2 + x_4^2) - f_2 x_4(x_1^2 - x_2^2) + \tilde{f}_2 x_3(x_1^2 - x_2^2) \\ & - g_2 x_2(x_3^2 - x_4^2) + \tilde{g}_2 x_1(x_3^2 - x_4^2) + 2f_2 x_1 x_2 x_3 + 2\tilde{f}_2 x_1 x_2 x_4 + 2g_2 x_1 x_3 x_4 + 2\tilde{g}_2 x_2 x_3 x_4,\end{aligned} \tag{44d}$$

where $a_1 = -\alpha_{11}/2$, $\tilde{a}_1 = -\sigma_1$, $\tilde{h}_1 = -\beta_{13}/2\omega$, $\tilde{i}_1 = \beta_{12}/4\omega$, $\tilde{j}_1 = \beta_{14}/4\omega$, $a_2 = -\alpha_{23}/2$, $\tilde{a}_2 = -\sigma_2$, $\tilde{h}_2 = -\beta_{21}/2\omega$, $\tilde{i}_2 = \beta_{22}/4\omega$, $\tilde{j}_2 = \beta_{24}/4\omega$. Other parameters can be found in the Appendix C.

The averaged Eq. (41) governs the amplitudes and phases of the approximate solutions. Taking into account Eqs. (35), (37) and (40), we obtain the approximate solutions of Eq. (26) as a sum of a contribution of order ε and a contribution of order ε^2

$$x(t) = \varepsilon \tilde{x}_1(t) + \varepsilon^2 \tilde{x}_2(t) + O(\varepsilon^3), \tag{45a}$$

$$\tilde{x}_1(t) = 2\rho(t)\cos(\omega t - \alpha(t)), \tag{45b}$$

$$\tilde{x}_2(t) = -\frac{2}{\omega^2}(\gamma_{15}\rho^2(t) + \gamma_{16}\rho(t)\chi(t)\cos(\alpha(t) - \beta(t)) + \gamma_{17}\chi^2(t)) + \frac{2}{3\omega^2}[\gamma_{15}\rho^2(t)\cos(2\omega t - 2\alpha(t)) + \gamma_{16}\rho(t)\chi(t)\cos(2\omega t - \alpha(t) - \beta(t)) + \gamma_{17}\chi^2(t)\cos(2\omega t - 2\beta(t))] \tag{45c}$$

and

$$y(t) = \varepsilon\tilde{y}_1(t) + \varepsilon^2\tilde{y}_2(t) + O(\varepsilon^3), \tag{46a}$$

$$\tilde{y}_1(t) = 2\chi(t)\cos(\omega t - \beta(t)), \tag{46b}$$

$$\tilde{y}_2(t) = -\frac{2}{\omega^2}(\gamma_{25}\rho^2(t) + \gamma_{26}\rho(t)\chi(t)\cos(\alpha(t) - \beta(t)) + \gamma_{27}\chi^2(t)) + \frac{2}{3\omega^2}[\gamma_{25}\rho^2(t)\cos(2\omega t - 2\alpha(t)) + \gamma_{26}\rho(t)\chi(t)\cos(2\omega t - \alpha(t) - \beta(t)) + \gamma_{27}\chi^2(t)\cos(2\omega t - 2\beta(t))]. \tag{46c}$$

The validity of the approximate solutions should be expected to be restricted on bounded intervals of variable τ, that is, time scale $t = O(1/\varepsilon^2)$. If we wish to construct the approximate

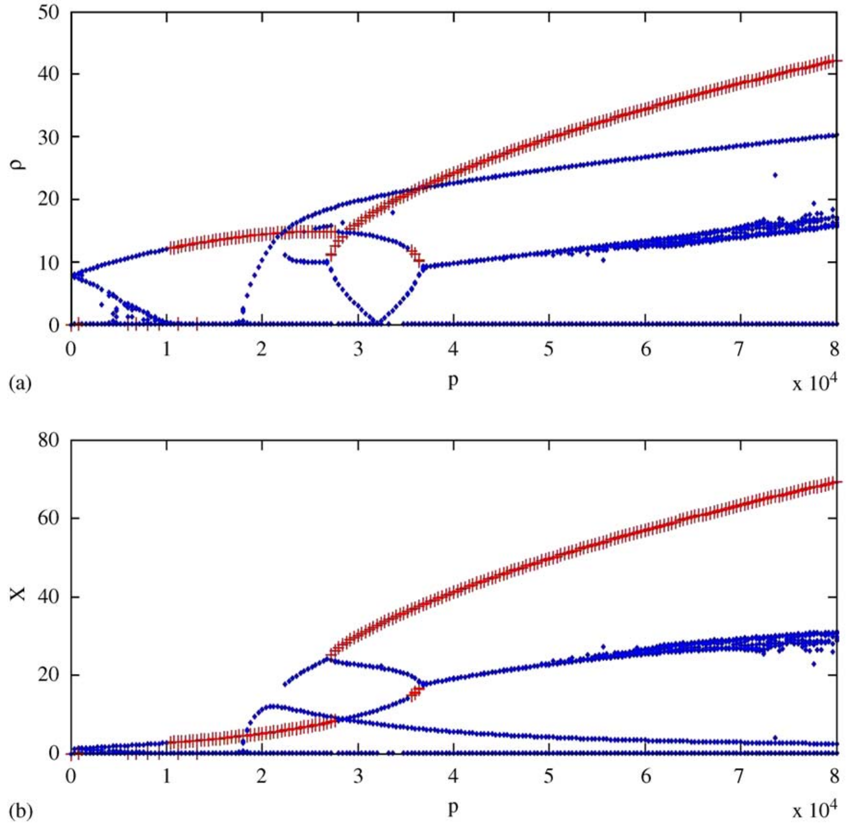

Fig. 2. Response–amplitude of excitation curves. The dots stand for unstable and plus sign for stable solutions ($\sigma_1 = 0, \sigma_2 = 0$).

solutions on larger intervals such that $\tau = O(1/\varepsilon)$, higher-order nonlinear terms will generally have the influence on the solutions.

4. Stability analysis of steady-state solutions

The fixed points of the averaged equation (41) correspond to the phase-locked periodic solutions of system (26). The stability of the solutions can be determined by investigating the characteristic equations. Therefore, letting $\mathrm{d}\rho/\mathrm{d}\tau = \mathrm{d}\chi/\mathrm{d}\tau = \mathrm{d}\alpha/\mathrm{d}\tau = \mathrm{d}\beta/\mathrm{d}\tau = 0$, we can obtain the algebraic equations determining the steady-state solutions of Eq. (26) as follows:

$$\begin{aligned} & a_1\rho + b_1\rho^3 + \tilde{c}_1\rho^2\chi\sin(\alpha-\beta) + e_1\chi^3\cos(\alpha-\beta) + \tilde{e}_1\chi^3\sin(\alpha-\beta) \\ & \quad + f_1\rho^2\chi\cos(\alpha-\beta) - \tilde{f}_1\rho^2\chi\sin(\alpha-\beta) + g_1\rho\chi^2\cos(2\alpha-2\beta) \\ & \qquad + \tilde{g}_1\rho\chi^2\sin(2\alpha-2\beta) + \tilde{h}_1\chi\sin(\alpha-\beta) + \tilde{i}_1\rho\sin 2\alpha + \tilde{j}_1\chi\sin(\alpha+\beta) = 0. \end{aligned} \tag{47a}$$

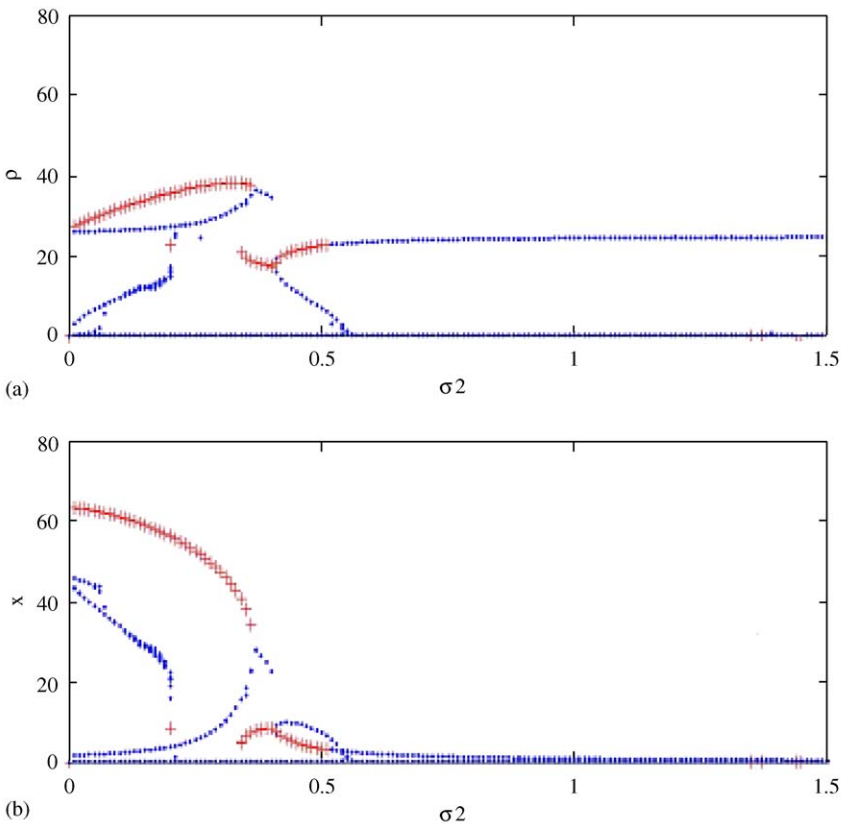

Fig. 3. Response–frequency of excitation curves. The dots stand for unstable and plus sign for stable solutions ($p = 61000, \sigma_1 = 0$).

$$\tilde{a}_1\rho + \tilde{b}_1\rho^3 + \tilde{c}_1\rho^2\chi\cos(\alpha-\beta) + \tilde{d}_1\rho\chi^2 - e_1\chi^3\sin(\alpha-\beta) + \tilde{e}_1\chi^3\cos(\alpha-\beta)$$
$$+ f_1\rho^2\chi\sin(\alpha-\beta) + \tilde{f}_1\rho^2\chi\cos(\alpha-\beta) - g_1\rho\chi^2\sin(2\alpha-2\beta)$$
$$+ \tilde{g}_1\rho\chi^2\cos(2\alpha-2\beta) + \tilde{h}_1\chi\cos(\alpha-\beta) + \tilde{i}_1\rho\cos(2\alpha) + \tilde{j}_1\chi\cos(\alpha+\beta) = 0, \quad (47b)$$

$$a_2\chi + b_2\rho^3\cos(\alpha-\beta) - \tilde{b}_2\rho^3\sin(\alpha-\beta) - \tilde{d}_2\rho\chi^2\sin(\alpha-\beta) + e_2\chi^3$$
$$+ f_2\rho^2\chi\cos(2\alpha-2\beta) - \tilde{f}_2\rho^2\chi\sin(2\alpha-2\beta) + g_2\rho\chi^2\cos(\alpha-\beta)$$
$$+ \tilde{g}_2\rho\chi^2\sin(\alpha-\beta) - \tilde{h}_2\rho\sin(\alpha-\beta) + \tilde{i}_2\rho\sin(\alpha+\beta) + \tilde{j}_2\chi\sin 2\beta = 0, \quad (47c)$$

$$\tilde{a}_2\chi + b_2\rho^3\sin(\alpha-\beta) + \tilde{b}_2\rho^3\cos(\alpha-\beta) + \tilde{c}_2\rho^2\chi + \tilde{d}_2\rho\chi^2\cos(\alpha-\beta) + \tilde{e}_2\chi^3$$
$$+ f_2\rho^2\chi\sin(2\alpha-2\beta) + \tilde{f}_2\rho^2\chi\cos(2\alpha-2\beta) - g_2\rho\chi^2\sin(\alpha-\beta)$$
$$+ \tilde{g}_2\rho\chi^2\cos(\alpha-\beta) + \tilde{h}_2\rho\cos(\alpha-\beta) + \tilde{i}_2\rho\cos(\alpha+\beta) + \tilde{j}_2\chi\cos(2\beta) = 0. \quad (47d)$$

In order to determine the stability of the steady state responses, let

$$\rho = \rho_0 + \delta\rho, \chi = \chi_0 + \delta\chi, \alpha = \alpha_0 + \delta\alpha, \beta = \beta_0 + \delta\beta, \quad (48)$$

where $(\rho_0, \chi_0, \alpha_0, \beta_0)$ are the solutions of Eq. (47), $\delta\rho, \delta\chi, \delta\alpha$ and $\delta\beta$ represent the perturbations to the steady-state responses. Substituting Eq. (48) into Eq. (41) and using Eq. (47), the parts of

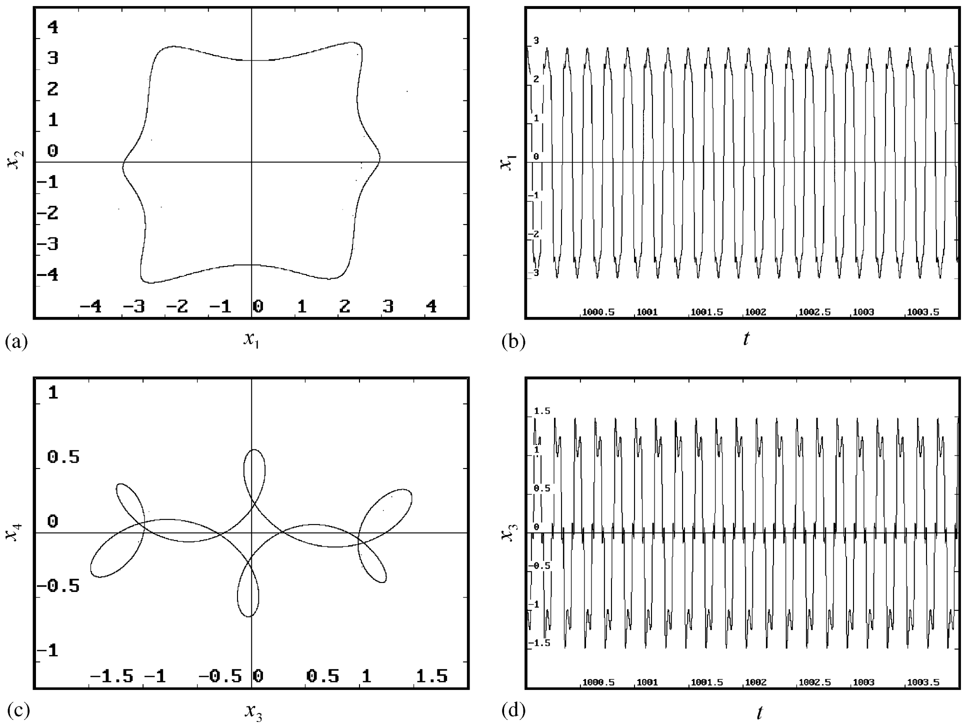

Fig. 4. The period motion for the first-order mode and period-7 motion for the second-order mode in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 4$, (a) the phase portrait on plane (x_1, x_2); (b) the waveform on plane (t, x_1); (c) phase portrait on plane (x_3, x_4) and (d) the waveform on plane (t, x_3).

the linearization for the resulting equations are

$$\frac{d(\delta\rho)}{dt} = C_{11}\delta\rho + C_{12}\delta\chi + C_{13}\delta\alpha + C_{14}\delta\beta, \tag{49a}$$

$$\frac{d(\delta\chi)}{dt} = C_{21}\delta\rho + C_{22}\delta\chi + C_{23}\delta\alpha + C_{24}\delta\beta, \tag{49b}$$

$$\frac{d(\delta\alpha)}{dt} = C_{31}\delta\rho + C_{32}\delta\chi + C_{33}\delta\alpha + C_{34}\delta\beta, \tag{49c}$$

$$\frac{d(\delta\beta)}{dt} = C_{41}\delta\rho + C_{42}\delta\chi + C_{43}\delta\alpha + C_{44}\delta\beta, \tag{49d}$$

where all coefficients $C_{ij}(i = 1, \ldots, 4; j = 1, \ldots, 4)$ are presented in the Appendix D.

The coefficient matrix of Eq. (49) is expressed by

$$A = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}. \tag{50}$$

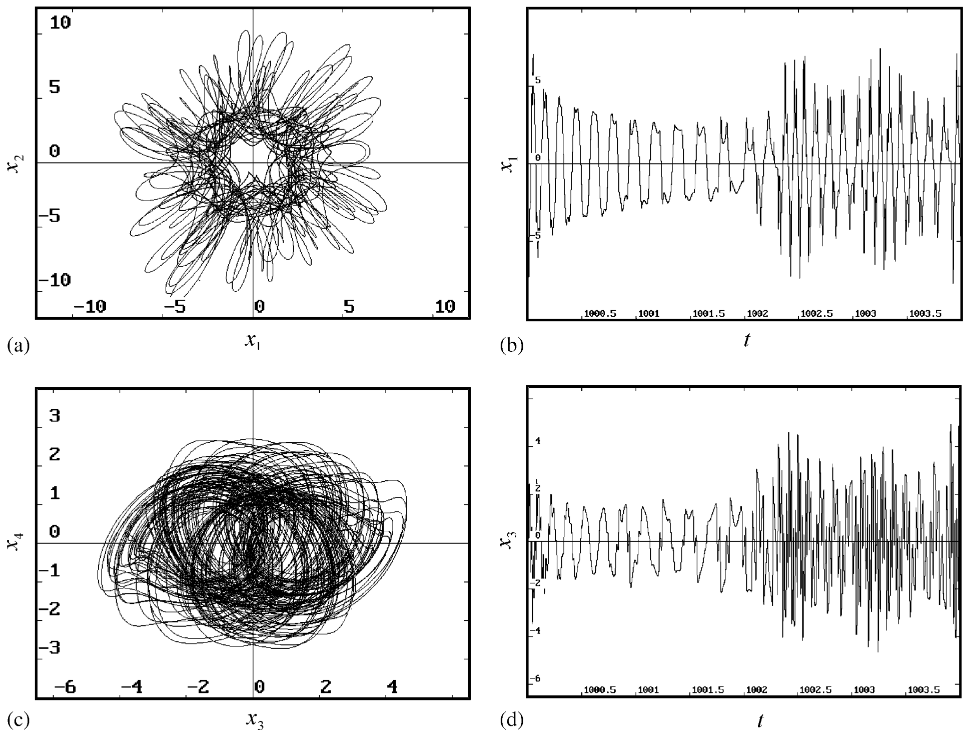

Fig. 5. Chaotic motion in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 10$.

Let

$$p = -\mathrm{tr}(A), \quad q = \det(A). \tag{51}$$

Then, the characteristic equation of the nontrivial steady-state solutions is of form

$$D(\lambda) = \lambda^2 + p\lambda + q = 0. \tag{52}$$

If the real parts of all eigenvalues are negative, the corresponding steady-state solutions are stable. Otherwise, these solutions are unstable. The Hopf bifurcation can cause the instability of the steady-state solutions since a pair of pure imaginary eigenvalues exists in Eq. (52). Furthermore, the complex eigenvalues with negative real parts for Eq. (52) change to those with positive real parts.

5. Numerical simulation of nonlinear responses

In this section, the steady-state nonlinear responses of the antisymmetric cross-ply laminated composite rectangular thin plate under parametric excitation is numerically investigated for the case of primary parametric resonance and 1:1 internal resonance.

Consider a Glass/Epoxy system [32] in which the parameters are chosen as follows:

$$E_1 = 7.8 \times 10^6\,\mathrm{lb/in^2}, \quad E_2 = 2.6 \times 10^6\,\mathrm{lb/in^2}, \quad v_{12} = 0.25, \quad Q_{11} = 7.966 \times 10^6\,\mathrm{lb/in^2},$$

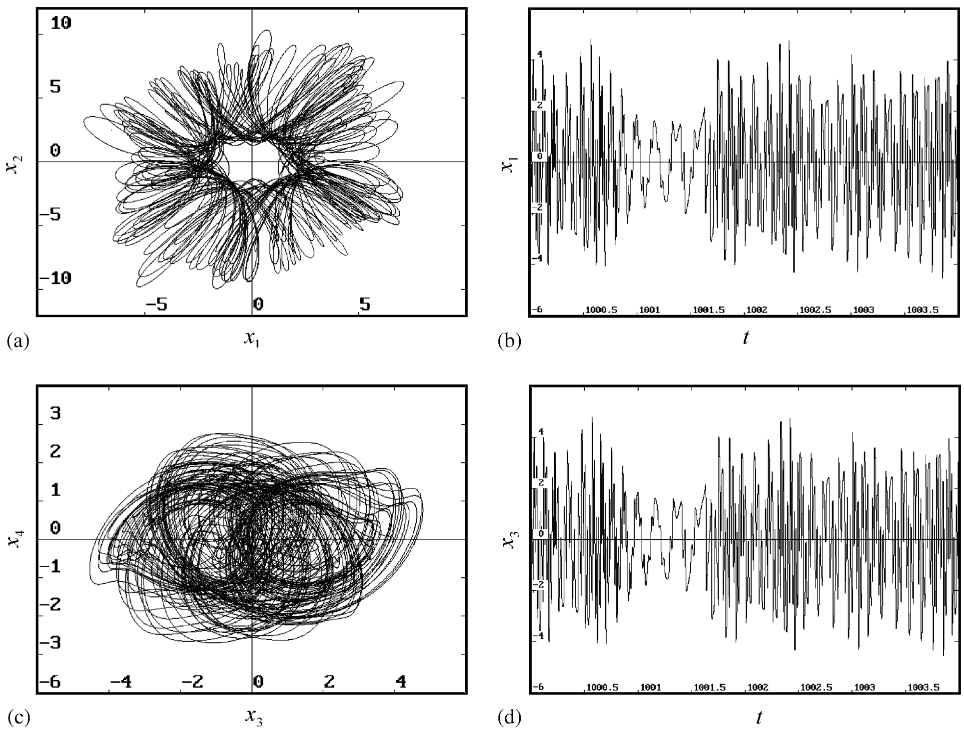

Fig. 6. Chaotic motion in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 20$.

$$Q_{12} = 0.6638 \times 10^6\,\mathrm{lb/in^2}, \quad Q_{22} = 2.655 \times 10^6\,\mathrm{lb/in^2}, \quad Q_{66} = 1.25 \times 10^6\,\mathrm{lb/in^2},$$

$$G_{12} = 1.25 \times 10^6\,\mathrm{lb/in^2}, \quad n = 4(0/90/0/90), \quad h = 1\,\mathrm{in}, \quad \rho = 5.1 \times 10^{-2}\,\mathrm{lb/in^2},$$

$$k = 0.1, \quad \eta = 0.2, \quad a = 100\,\mathrm{in}, \quad b = 30\,\mathrm{in}, \quad N_0 = 10\,\mathrm{lb}.$$

Using the aforementioned parametric values, the damping and stiffness coefficients can be obtained by Eq. (B.1) in Appendix B. In the following numerical study, we only analyze the influence of the parametric excitation and frequency on nonlinear responses of the antisymmetric cross-ply laminated composite rectangular thin plate.

Using Eq. (47), the numerical results are obtained, as shown in Figs. 2 and 3. Fig. 2 indicates that the amplitudes of the first two modes are changed as the amplitude of parametric excitation $N_1 = p$ changes, where $\sigma_1 = 0$ and $\sigma_2 = 0$. The dot and plus sign represent the unstable and stable solutions, respectively. It is found that the trivial solution is always stable, which means that an equilibrium state of the antisymmetric cross-ply laminated composite rectangular thin plate under parametric excitation always exists. However, as parametric excitation N_1 increases from zero to σ_A, the nontrivial solutions change to unstable. When $N_1 > \sigma_A$, one of the nontrivial solutions becomes stable. Fig. 2b gives another situation in which there exist not any stable nontrivial solutions for small parametric excitation. When the excitation amplitude N_1 is larger than a critical value σ_A, there is a region where only one stable steady state solution appears.

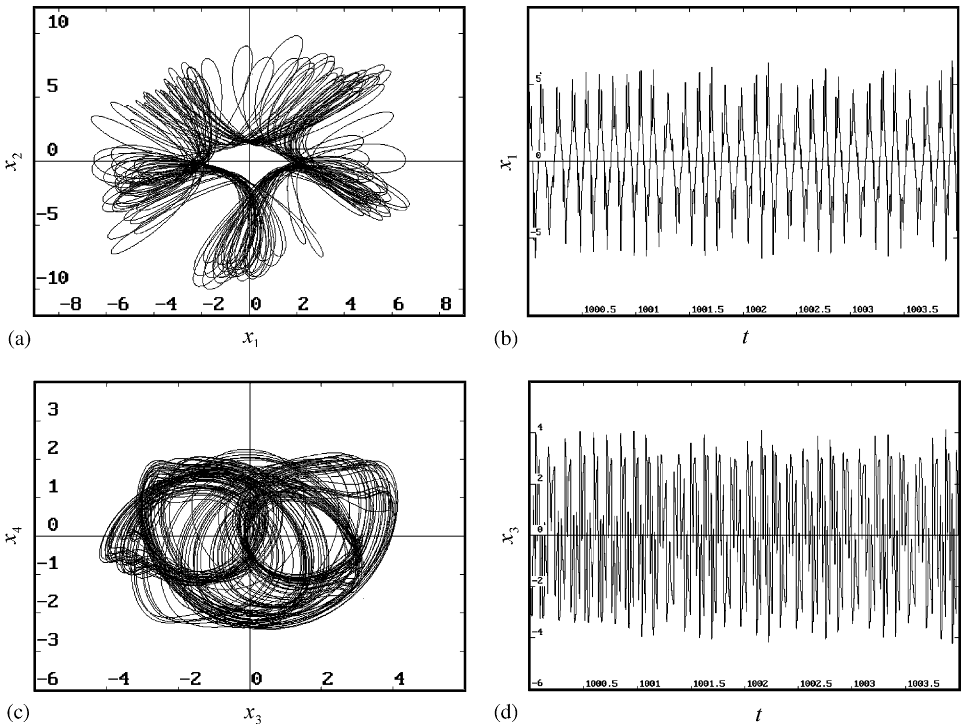

Fig. 7. Chaotic motion in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 90$.

In Fig. 3, the frequency–response curves are given for the case $N_1 = 61\,000$ and $\sigma_1 = 0$. Similar to the aforementioned situation, only one stable-steady solution appears when σ_2 increases from zero. There exist not any stable solutions when σ_2 is greater than σ_B.

Comparing with the numerical simulations, the asymptotic perturbation method gives a quite accurate prediction on the steady-state periodic responses when the amplitude of the parametric excitation N_1 is small. Nevertheless, the solutions obtained by using the analytical method are smaller than the exact ones when the parametric excitation N_1 is rather large. The differences are caused by the introduction of the small parameter ε in the application of the asymptotic perturbation method.

6. Numerical simulation of periodic and chaotic motions

We choose averaged Eq. (44) to do numerical simulation of the periodic and chaotic motions. Numerical approach through a computer software *Dynamics* [34] is utilized to explore the existence of the periodic and chaotic motions in the antisymmetric cross-ply laminated composite rectangular thin plate under parametric excitation.

We choose that parametric excitation β_{12} as controlling parameter when the periodic and chaotic responses in the antisymmetric cross-ply laminated composite rectangular thin plate are

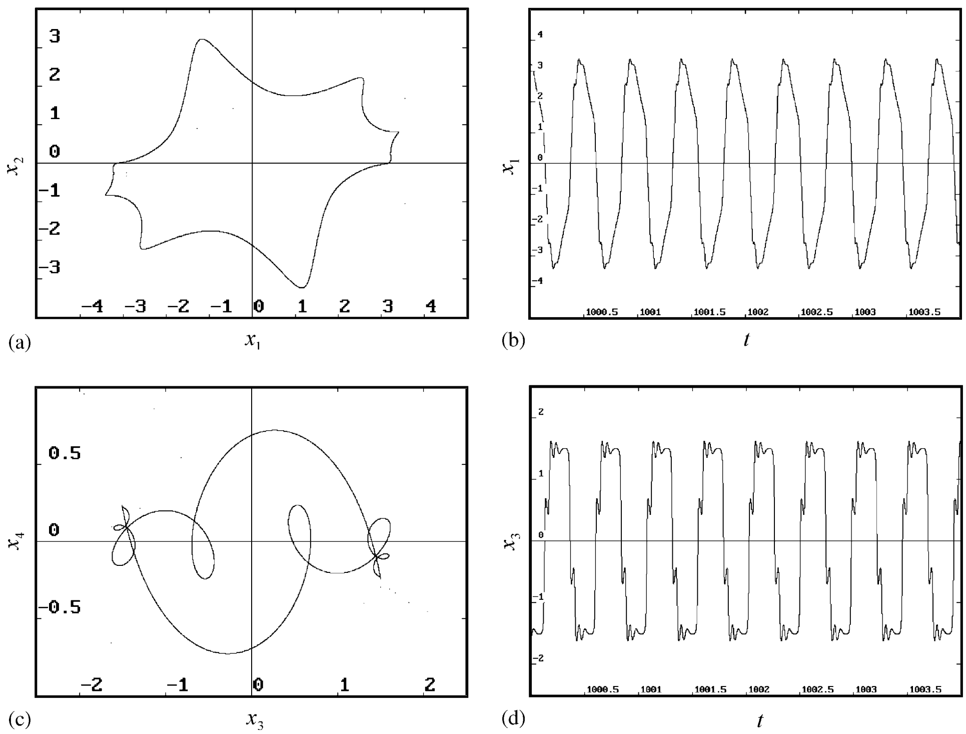

Fig. 8. The period motion for the first-order mode and period-7 motion for the second-order mode in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 132$.

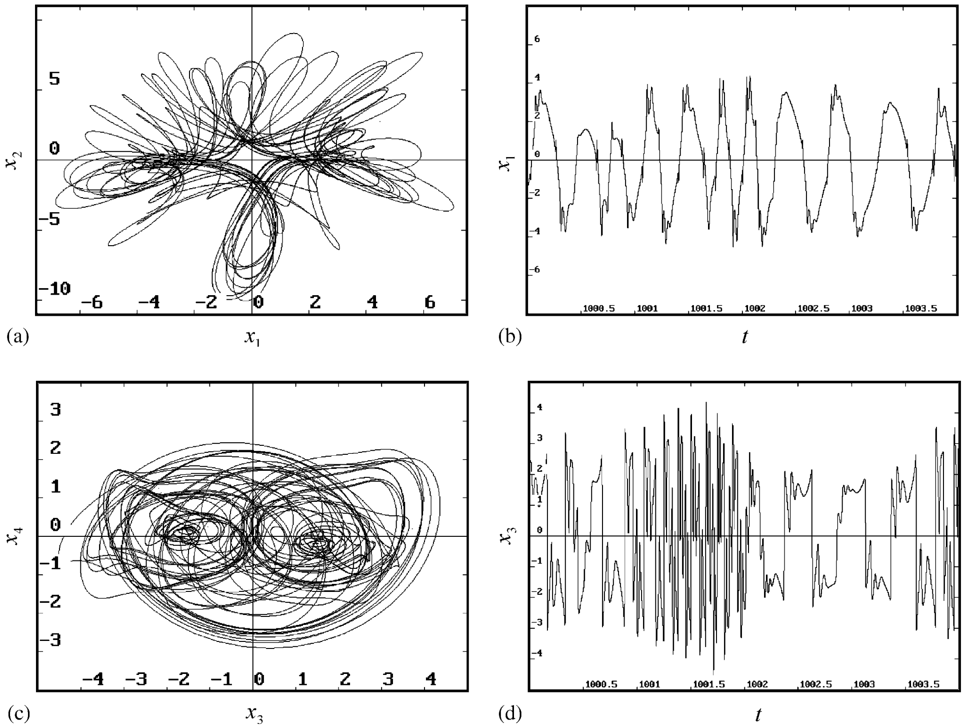

Fig. 9. The multipulse chaotic motion in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 138$.

investigated. In the following investigation, we will indicate that the motions of the antisymmetric cross-ply laminated composite rectangular thin plate can be changed from the periodic motion to the chaotic motion, then to the periodic motion as the parametric excitation β_{12} changes, as shown in Figs. 4–12.

Fig. 4 demonstrates the existence of the period motion for the first-order mode and period-7 motion for the second-order mode in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 4$. Other parameters and initial conditions are chosen as $\alpha_{11} = \alpha_{23} = 0.16$, $\sigma_1 = 3.26$, $\sigma_2 = 1.57$, $\omega = 1$, $\beta_{14} = 20$, $\beta_{22} = 60$, $\beta_{24} = 80$, $\tilde{h}_1 = -1.4$, $\tilde{h}_2 = -1.6$, $b_1 = -0.021$, $b_2 = -0.02$, $\tilde{b}_1 = 2.1$, $\tilde{b}_2 = 2.7$, $\tilde{c}_1 = 16.1$, $\tilde{c}_2 = 11.93$, $\tilde{d}_1 = 3.4$, $\tilde{d}_2 = 2.9$, $e_1 = -1.5$, $e_2 = -1.06$, $\tilde{e}_1 = 3.8$, $\tilde{e}_2 = 2.14$, $f_1 = -1.61$, $f_2 = -1.08$, $\tilde{f}_1 = -5.3$, $\tilde{f}_2 = -1.22$, $g_1 = -1.41$, $g_2 = 1.18$, $\tilde{g}_1 = -3.2$, $\tilde{g}_2 = 2.11$, $x_{10} = 0.12$, $x_{20} = 0.05$, $x_{30} = 0.325$, $x_{40} = 0.16$. Figs. 4a–d, respectively, represent the phase portraits on the planes (x_1, x_2), (x_3, x_4) and the waveforms on the planes (t, x_1), (t, x_3). When the parametric excitation β_{12}, is respectively, increased to $\beta_{12} = 10$, $\beta_{12} = 20$ and $\beta_{12} = 90$, the chaotic motions of the antisymmetric cross-ply laminated composite rectangular thin plate occur, as shown in Figs. 5–7. In these cases, the chosen parameters and initial conditions are same as those in Fig. 4. It is found from Figs. 5–7 that there exists a large difference between the phase portrait on the plane (x_1, x_2) and the phase portrait on the plane (x_3, x_4).

When the parametric excitation β_{12} is changed to $\beta_{12} = 132$, it is also found that the period motion for the first-order mode and period-7 motion for the second-order mode exist in the

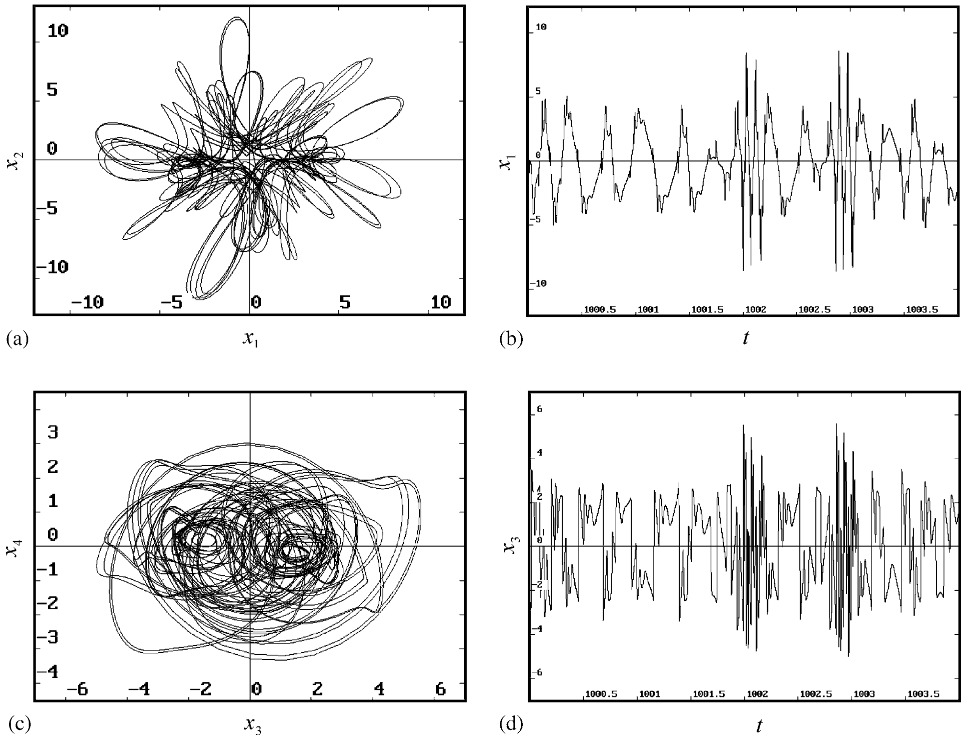

Fig. 10. The multipulse chaotic motion in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 144$.

antisymmetric cross-ply laminated composite rectangular thin plate, as shown in Fig. 8. In this case, the chosen parameters and initial conditions are same as those in Fig. 4. Figs. 9–11 indicate the existence of the chaotic motions of the antisymmetric cross-ply laminated composite rectangular thin plate when the parametric excitation β_{12}, respectively, is $\beta_{12} = 138$, $\beta_{12} = 144$ and $\beta_{12} = 150$. In these cases, the chosen parameters and initial conditions are same as those in Fig. 4. It is observed that the multipulse chaotic motions exist in Figs. 9–11. Continuously increasing the parametric excitation to $\beta_{12} = 156$, it is found that the multipulse periodic motion occurs, as shown in Fig. 12.

It is seen from Figs. 9–11 that the shape of the chaotic motion for the first-order mode is different from that for the second-order mode. The existence on the multipulse orbits in the antisymmetric cross-ply laminated composite rectangular thin plate is also observed in Figs. 9–12.

7. Conclusions

In this paper, we investigate nonlinear oscillations and chaotic dynamics of a four-edges simply supported antisymmetric cross-ply laminated composite rectangular thin plate subjected to in-plane load. In the aforementioned investigations, material, geometrical and damping nonlinearities are simultaneously considered. The governing partial differential equations of motion for

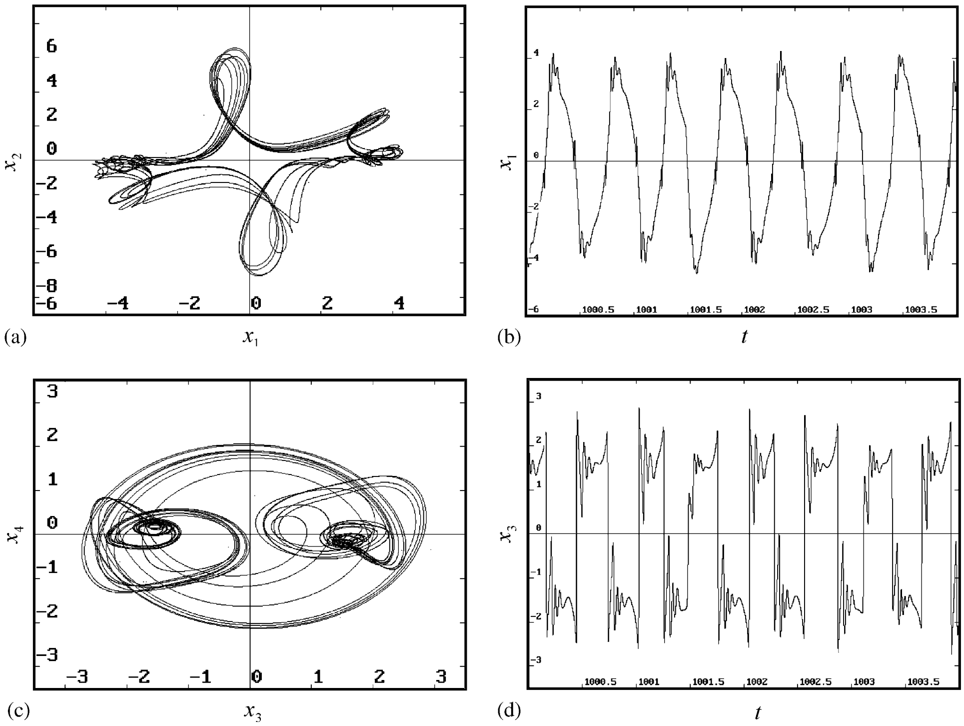

Fig. 11. The multipulse chaotic motion in the antisymmetric cross-ply laminated composite rectangular thin plate when $\beta_{12} = 150$.

the antisymmetric cross-ply laminated composite rectangular thin plate is first derived by using the von Karman-type nonlinear plate theory. The Galerkin method is used to reduce the partial differential equations to a two dof nonlinear system including the quadratic, cubic nonlinear terms and parametric excitations. The asymptotic perturbation method is utilized to obtain the averaged equations of the system under parametric excitation for the first time. Based on the averaged equations, the steady-state nonlinear responses and their stabilities are determined by using numerical approach. The relation between the steady-state nonlinear responses and the amplitude and frequency of parametric excitation is obtained. Under the certain conditions, the antisymmetric cross-ply laminated composite rectangular thin plate may have two steady-state nonzero solutions in which the jumping phenomenon occurs. Numerical simulation is also used to discover the periodic and chaotic motions in the antisymmetric cross-ply laminated composite rectangular thin plate. It is observed from the numerical results that the multipulse orbits exist in the antisymmetric cross-ply laminated composite rectangular thin plate.

In Refs. [9,10], Ganapathi et al. used the eight-noded finite element method to study the nonlinear instability behavior of angle-ply laminated composite plate subjected to periodic in-plane load. In this paper, we only analyze nonlinear oscillations and chaotic dynamics of a four-edges simply supported antisymmetric cross-ply laminated composite rectangular thin plate subjected to in-plane load by using the asymptotic perturbation method developed by Maccari [24–28]. The problem considered by this paper is different from one considered

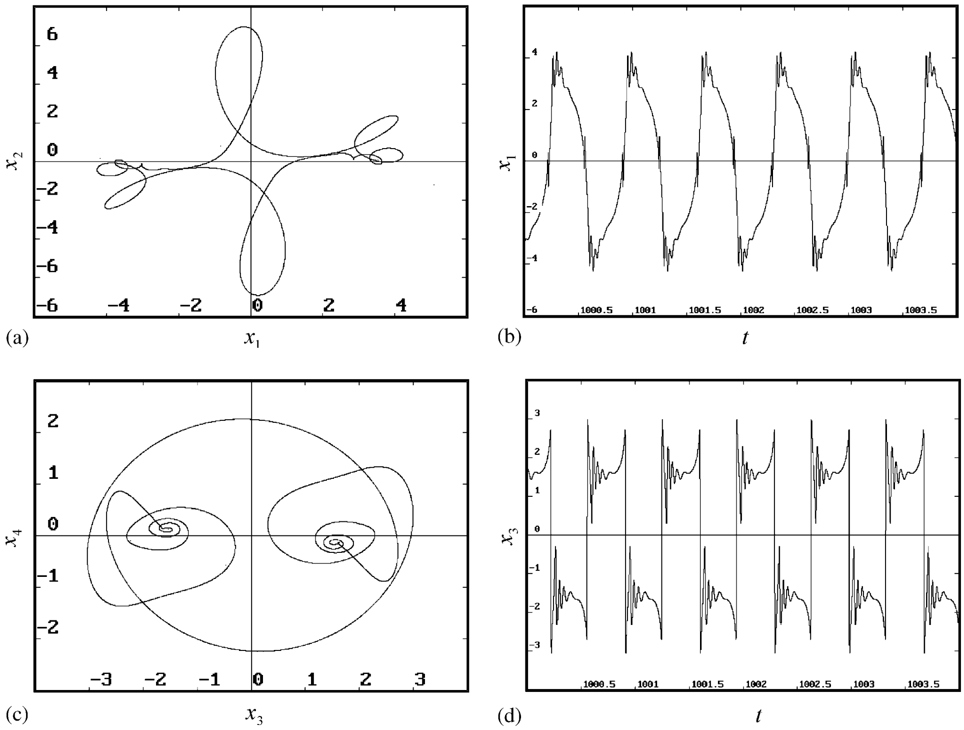

Fig. 12. The multipulse periodic motion in the antisymmetric cross-ply laminated composite rectangular thin when $\beta_{12} = 156$.

by Ganapathi et al. [9,10] although the quadratic terms are included in the equations of transverse motion.

Acknowledgements

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Appendix A

The coefficients μ_{ij} $(i = 1, 2; j = 1, \ldots, 4)$, h_{ij} $(i,j = 1, 2)$ and Ω_{ij} $(i,j = 1, 2)$ presented in Eq. (21) are as follows:

$$\mu_{11} = \frac{3}{8}\frac{\pi^4 k}{\eta a^3 \rho h}\left(1 + \frac{A_{12}}{A_{11}}\alpha^2\right), \quad \mu_{12} = \frac{1}{4}\frac{\pi^4 k}{\eta a^3 \rho h}\frac{A_{12}}{A_{11}}\alpha^2, \quad \mu_{13} = \frac{3}{2}\frac{\pi^4 k}{\eta a^3 \rho h}\left(1 + \frac{A_{12}}{A_{11}}\alpha^2\right),$$

$$\mu_{14} = \frac{\pi^4 k}{\eta a^3 \rho h}\frac{A_{12}}{A_{11}}\alpha^2, \quad h_{11} = -\frac{\pi^2}{2a^4\rho h}\frac{g_2}{\Omega_{11}^2}N_1, \quad h_{12} = \frac{2\pi^2}{3a^2\rho h\Omega_{12}^2}\frac{A_{12}}{A_{11}}\alpha^2 N_1,$$

$$\Omega_{11}^2 = \frac{\pi^4}{a^4\rho h}\left\{\left[8D_{11}(1+\alpha^4) + \frac{16}{3}\alpha^2(D_{12}+2D_{66}) - \frac{16}{3}\frac{B_{11}^2}{A_{11}}(1+\alpha^4) + \frac{k_0}{\pi^2 k_1}\right] - \frac{2a^2}{\pi^2}\left(1+\frac{A_{12}}{A_{11}}\alpha^2\right)N_0\right\},$$

$$k_0 = -\frac{8}{3}B_{11}^2\pi^2[A_{66}(\alpha^6+\alpha^{-2}) + A_{11}(1+\alpha^4) + 2(A_{12}+A_{66})\alpha^2],$$

$$k_1 = (A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) + A_{11}A_{66}(\alpha^2+\alpha^{-2}),$$

$$\Omega_{12}^2 = \frac{4}{3}\frac{\pi^2}{a^4\rho h}\left[4\pi^2\alpha^4\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right) + 2\pi^2\alpha^2\frac{B_{11}A_{66}}{A_{11}} - a^2\alpha^2\frac{A_{12}}{A_{11}}N_0\right],$$

$$\mu_{21} = \frac{1}{4}\frac{\pi^4 k}{\eta a^3\rho h}\left(\frac{A_{12}}{A_{11}}\alpha^2\right), \quad \mu_{22} = \frac{3}{8}\frac{\pi^4 k}{\eta a^3\rho h}\left(4+\frac{A_{12}}{A_{11}}\alpha^2\right), \quad \mu_{23} = \frac{\pi^4 k}{\eta a^3\rho h}\left(\frac{A_{12}}{A_{11}}\alpha^2\right),$$

$$\mu_{24} = \frac{3}{2}\frac{\pi^4 k}{\eta a^3\rho h}\left(4+\frac{A_{12}}{A_{11}}\alpha^2\right), \quad \Omega_{21}^2 = \frac{4}{3}\frac{\pi^2}{a^4\rho h}\left[4\pi^2\alpha^4\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right) - a^2\alpha^2\frac{A_{12}}{A_{11}}N_0\right],$$

$$\Omega_{22}^2 = \frac{2\pi^4}{a^4\rho h}\left\{\frac{4}{3}\left[\left(3D_{11} - 2\frac{B_{11}^2}{A_{11}}\right)(16+\alpha^4) + \left(8D_{22}+16D_{66} - \frac{B_{11}A_{66}}{A_{11}}\right)\alpha^2 - \frac{k_2}{k_3}\right] - \frac{a^2}{\pi^2}\left(4+\frac{A_{12}}{A_{11}}\alpha^2\right)N_0\right\},$$

$$k_2 = A_{66}B_{11}^2(\alpha^6 + 256\alpha^{-2}) + B_{11}(A_{66}^2 + 4A_{11}B_{11})\alpha^4 + 4B_{11}(A_{11}A_{66} + 8A_{66}B_{11} + 8A_{12}B_{11})\alpha^2 + 16B_{11}(A_{12}A_{66} + A_{66}^2 + 4A_{11}B_{11}),$$

$$k_3 = 4(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) + A_{11}A_{66}(\alpha^2 + 16\alpha^{-2}),$$

$$h_{21} = \frac{2\pi^2}{3a^2\rho h\Omega_{21}^2}\frac{A_{12}}{A_{11}}\alpha^2 N_1, \quad h_{22} = \frac{\pi^2}{a^2\rho h\Omega_{22}^2}\left(4+\frac{A_{12}}{A_{11}}\alpha^2\right)N_1, \tag{A.1}$$

where $\alpha = a/b$ is the ratio between the width and length of the plate.

The coefficients φ_{1i} $(i = 1, \ldots, 7)$ and φ_{2i} $(i = 1, \ldots, 7)$ in Eq. (21) are given as

$$\varphi_{11} = \frac{2b^2B_{11}}{3a^2m_2}(2A_{12}^2 - 2A_{11}^2 + A_{12}A_{66}) + \frac{2a^2B_{11}}{3b^2m_2}(2A_{11}^2 - 2A_{12}^2 - A_{12}A_{66}) + \frac{4A_{11}A_{66}B_{11}}{3m_2}\left(\frac{a^4}{b^4} - \frac{b^4}{a^4}\right) + \frac{4}{3}\frac{\pi^4 B_{11}}{\rho h}\left(\frac{1}{a^4} - \frac{1}{b^4}\right),$$

$$\begin{aligned}
\varphi_{12} = {} & \frac{1}{24m_2}(2A_{12}A_{66}^2 + A_{12}^2A_{66} - A_{11}^2A_{66} + 104A_{11}A_{66}B_{11}) \\
& + \frac{2A_{66}}{3m_1}(3A_{12}A_{66} - 2A_{11}^2 + 2A_{12}^2 - 4A_{11}B_{11}) + \frac{1}{24}\frac{\pi^4}{a^4\rho hA_{11}}(17A_{12}A_{66} - 160A_{11}B_{11}) \\
& + \frac{b^2}{24a^2m_2}(96A_{11}^2B_{11} - 96A_{12}^2B_{11} - 144A_{12}A_{66}B_{11} - A_{11}A_{66}^2) \\
& + \frac{8b^2}{3a^2m_1}(4A_{11}^2B_{11} - 4A_{12}^2B_{11} - A_{11}A_{66}^2 - 4A_{12}A_{66}B_{11}) \\
& + \frac{4b^4A_{11}A_{66}B_{11}}{a^4m_2} + \frac{128b^4A_{11}A_{66}B_{11}}{3a^4m_1} \\
& + \frac{a^2}{24b^2m_2}(8A_{11}^2B_{11} - 8A_{12}^2B_{11} - A_{11}A_{66}^2 - 64A_{12}A_{66}B_{11}) + \frac{a^4A_{11}A_{66}B_{11}}{3b^4m_2} - \frac{a^4A_{11}A_{66}B_{11}}{3b^4m_1} \\
& + \frac{a^2}{3b^2m_1}(4A_{12}^2B_{11} - 4A_{11}^2B_{11} - A_{11}A_{66}^2 + 6A_{12}A_{66}B_{11}) \\
& + \frac{\pi^4}{8a^2b^2\rho hA_{11}}(9A_{11}A_{66} + 16A_{12}B_{11}),
\end{aligned}$$

$$\begin{aligned}
\varphi_{13} = {} & \frac{8A_{66}}{3m_1}(A_{11}^2 - 4A_{11}B_{11} - A_{12}^2 - A_{12}A_{66}) + \frac{A_{66}}{12m_2}(A_{11}^2 - A_{12}A_{66} - A_{12}^2) + \frac{a^2A_{66}^2A_{11}}{12b^2m_2} \\
& + \frac{128b^2B_{11}}{3a^2m_1}(3A_{12}A_{66} - 2A_{11}^2 + 2A_{12}^2) + \frac{2a^2}{3b^2m_1}(A_{11}A_{66}^2 + 4A_{11}^2B_{11} - 4A_{12}^2B_{11} \\
& - 4A_{12}A_{66}B_{11}) - \frac{1024b^4A_{11}A_{66}B_{11}}{3a^4m_1} + \frac{2a^4A_{11}A_{66}B_{11}}{3b^4m_1} + \frac{64}{3}\frac{\pi^4B_{11}}{a^4\rho h} - \frac{2}{3}\frac{\pi^4B_{11}}{b^4\rho h} \\
& + \frac{A_{66}\pi^4}{12a^2b^2\rho hA_{11}}(32B_{11} + 9A_{11}),
\end{aligned}$$

$$\begin{aligned}
\varphi_{14} = {} & \frac{1}{12m_1}(6A_{11}^2A_{66} - 9A_{12}^2A_{66} - 8A_{12}A_{66}^2 - 2A_{12}^3 + 2A_{11}^2A_{12}) \\
& + \frac{1}{12m_3}(6A_{11}^2A_{66} - 2A_{12}^3 - 8A_{12}A_{66}^2 + 2A_{11}^2A_{12} - 9A_{12}^2A_{66}) + \frac{4b^4A_{11}^2A_{66}}{3a^4m_1} + \frac{b^4A_{11}^2A_{66}}{192a^4m_2} \\
& + \frac{1}{96m_2}(67A_{11}^2A_{66} - 68A_{12}A_{66}^2 - 68A_{12}^2A_{66} + A_{11}^2A_{12} - A_{12}^3) + \frac{4a^4A_{11}^2A_{66}}{3b^4m_3} + \frac{a^4A_{11}^2A_{66}}{192b^4m_2} \\
& + \frac{A_{11}}{3}\left(\frac{b^2}{a^2m_1} + \frac{a^2}{b^2m_3}\right)(4A_{66}^2 - A_{12}^2 + A_{11}^2) + \frac{A_{11}A_{66}}{24}\left(\frac{b^2}{a^2m_3} + \frac{a^2}{b^2m_1}\right)(2A_{66} + A_{12}) \\
& + \frac{b^2A_{11}}{192a^2m_2}(68A_{66}^2 - A_{12}^2 + A_{11}^2) + \frac{a^2A_{11}}{192b^2m_2}(A_{11}^2 - A_{12}^2 - 68A_{66}^2)
\end{aligned}$$

$$- \frac{\pi^4}{96a^2b^2\rho h}(9A_{12} + 50A_{66}) + \frac{17\pi^4}{192a^4\rho h A_{11}}(A_{11}^2 - 2A_{12}^2) + \frac{\pi^4}{192b^4\rho h A_{11}}(53A_{11}^2 - 70A_{12}^2),$$

$$\begin{aligned}\varphi_{15} = {} & \frac{a^4A_{11}^2A_{66}}{b^4m_3} - \frac{a^4A_{11}^2A_{66}}{32b^4m_2} - \frac{a^2}{12b^2m_1}(8A_{66}^2A_{11} + A_{11}A_{12}A_{66}) + \frac{\pi^4}{96a^2b^2\rho h}(100A_{66} - 51A_{12}) \\ & + \frac{a^2}{24b^2m_3}(6A_{11}A_{12}^2 - 16A_{66}^2A_{11} + 6A_{11}^3 - 4A_{11}A_{12}A_{66}) - \frac{32b^4A_{11}^2A_{66}}{3a^4m_1} + \frac{b^4A_{11}^2A_{66}}{24a^4m_3} \\ & + \frac{a^2}{96b^2m_2}(3A_{11}A_{12}^2 - 32A_{66}^2A_{11} - 3A_{11}^3 + 63A_{11}A_{12}A_{66}) + \frac{5b^4A_{11}^2A_{66}}{8a^4m_2} \\ & + \frac{b^2}{3a^2m_1}(8A_{11}A_{12}^2 - 8A_{11}^3 + 12A_{12}A_{11}A_{66} - 32A_{66}^2A_{11}) + \frac{\pi^4}{32b^4\rho h A_{11}}(13A_{11}^2 - 14A_{12}^2) \\ & + \frac{b^2}{48a^2m_3}(8A_{11}^3 - 2A_{66}^2A_{11} - 15A_{12}A_{11}A_{66} - 8A_{11}A_{12}^2) \\ & + \frac{b^2}{96a^2m_2}(60A_{11}^3 - 63A_{11}A_{12}A_{66} - 60A_{11}A_{12}^2 - 32A_{11}A_{66}^2) \\ & + \frac{1}{3m_1}(16A_{12}A_{66}^2 - 13A_{11}^2A_{66} + 13A_{12}^2A_{66} + A_{12}^3 - A_{11}^2A_{12}) \\ & + \frac{1}{48m_3}(6A_{11}^2A_{66} - 4A_{12}^3 + 16A_{12}A_{66}^2 + 4A_{11}^2A_{12} + 21A_{12}^2A_{66}) \\ & + \frac{1}{96m_2}(57A_{11}^2A_{12} - 57A_{12}^3 + 64A_{12}A_{66}^2 - 47A_{11}^2A_{66} - 10A_{12}^2A_{66}),\end{aligned}$$

$$\begin{aligned}\varphi_{16} = {} & \frac{1}{96m_4}(642A_{11}^2A_{66} - 865A_{12}^2A_{66} - 864A_{12}A_{66}^2 - 216A_{12}^3 + 216A_{11}^2A_{12}) + \frac{19\pi^4A_{11}}{48b^2\rho h} \\ & + \frac{1}{24m_5}(165A_{11}^2A_{66} - 205A_{12}^2A_{66} + 36A_{11}^2A_{12} - 216A_{12}A_{66}^2 - 36A_{12}^3) - \frac{7A_{12}^2\pi^4}{16b^4\rho h A_{11}} \\ & + \frac{1}{96m_3}(8A_{12}^3 + 32A_{12}A_{66}^2 - 94A_{11}^2A_{66} - 8A_{11}^2A_{12} - 17A_{12}^2A_{66}) + \frac{108b^4A_{11}^2A_{66}}{a^4m_5} \\ & + \frac{1}{48m_2}(15A_{11}^2A_{12} - 15A_{12}^3 - 37A_{11}^2A_{66} - 12A_{12}A_{66}^2 - 56A_{12}^2A_{66}) + \frac{27b^4A_{11}^2A_{66}}{4a^4m_4} \\ & + \frac{1}{3m_1}(22A_{11}^2A_{66} - 30A_{12}A_{66}^2 - 22A_{12}^2A_{66}) - \frac{\pi^4}{48a^2b^2\rho h}(25A_{12} + 124A_{66}) \\ & + \frac{b^2}{16a^2m_4}(162A_{11}A_{66}^2 - 15A_{11}A_{12}A_{66} + 48A_{11}^3 - 48A_{11}A_{12}^2)\end{aligned}$$

$$
\begin{aligned}
&+\frac{a^2}{6b^2m_5}(3A_{11}A_{66}^2+A_{11}A_{12}A_{66})\\
&+\frac{b^2}{2a^2m_5}(81A_{11}A_{66}^2-21A_{11}A_{12}A_{66}+24A_{11}^3-24A_{11}A_{12}^2)\\
&+\frac{17\pi^4}{24a^4\rho hA_{11}}(A_{11}^2-A_{12}A_{66}-A_{12}^2)\\
&+\frac{b^2}{48a^2m_3}(31A_{11}A_{12}A_{66}-2A_{11}A_{66}^2-16A_{11}^3+16A_{11}A_{12}^2)\\
&-\frac{1}{12}\frac{b^4A_{11}^2A_{66}}{a^4m_3}-\frac{4}{3}\frac{b^4A_{11}^2A_{66}}{a^4m_2}\\
&+\frac{b^2}{48a^2m_2}(64A_{11}A_{12}^2+6A_{11}A_{66}^2+143A_{11}A_{12}A_{66}-64A_{11}^3)+\frac{56b^2A_{11}A_{66}^2}{3a^2m_1}\\
&+\frac{1}{8}\frac{a^2}{b^2m_4}(3A_{11}^3-3A_{11}A_{12}^2+16A_{11}A_{66}^2+2A_{11}A_{12}A_{66})+\frac{a^4A_{11}^2A_{66}}{6b^4m_4}\\
&+\frac{a^2}{24b^2m_3}(A_{11}^3-A_{11}A_{12}^2-16A_{11}A_{66}^2-10A_{11}A_{12}A_{66})+\frac{a^4A_{11}^2A_{66}}{6b^4m_3}+\frac{a^4A_{11}^2A_{66}}{48b^4m_2}\\
&+\frac{a^2}{48b^2m_2}(A_{11}^3-A_{11}A_{12}^2+6A_{11}A_{66}^2+13A_{11}A_{12}A_{66})+\frac{4a^2A_{66}^2A_{11}}{3b^2m_1},
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{17}=\;&\frac{64b^2A_{66}^2A_{11}}{3a^2m_1}+\frac{2a^2A_{66}^2A_{11}}{3b^2m_1}+\frac{a^4A_{11}^2A_{66}}{24b^4m_2}\\
&+\frac{a^2}{24b^2m_2}(A_{11}^3+6A_{11}A_{66}^2-A_{11}A_{12}^2+2A_{11}A_{12}A_{66})\\
&+\frac{1}{24m_2}(8A_{11}^2A_{66}+4A_{11}^2A_{12}-4A_{12}^3-14A_{12}A_{66}^2-15A_{12}^2A_{66})\\
&+\frac{1}{3m_1}(16A_{11}^2A_{66}-24A_{12}A_{66}^2-16A_{12}^2A_{66})+\frac{1}{24}\frac{\pi^4}{b^4\rho hA_{11}}(4A_{11}^2-5A_{12}^2)\\
&-\frac{\pi^4}{12a^2b^2\rho h}(2A_{12}+29A_{66})+\frac{b^2}{6a^2m_2}(2A_{11}+A_{11}A_{12}A_{66}),
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{21}=\;&\frac{b^2B_{11}}{3a^2m_2}(2A_{12}^2+7A_{12}A_{66}-2A_{11}^2)+\frac{a^2B_{11}}{3b^2m_2}(A_{11}^2-5A_{12}A_{66}-A_{12}^2)-\frac{A_{66}A_{11}B_{11}}{3m_2}\\
&+\frac{16b^2B_{11}}{3a^2m_1}(A_{12}^2-A_{11}^2+A_{12}A_{66})-\frac{a^2A_{12}A_{66}B_{11}}{3b^2m_1}-\frac{2b^4A_{11}A_{66}B_{11}}{3a^4m_2}\\
&+\frac{64b^4B_{11}A_{66}A_{11}}{3a^2m_1}+\frac{a^4A_{11}A_{66}B_{11}}{3b^4m_2}-\frac{4A_{66}A_{11}B_{11}}{3m_1}+\frac{2\pi^4B_{11}}{a^4\rho h}-\frac{1}{3}\frac{\pi^4B_{11}}{b^4\rho h}+\frac{\pi^4B_{11}A_{12}}{a^2b^2\rho hA_{11}},
\end{aligned}
$$

$$\varphi_{22} = \frac{8A_{66}}{3m_1}(4A_{11}B_{11} + A_{11}^2 - A_{12}^2 - A_{12}A_{66}) + \frac{A_{66}}{12m_2}(A_{11}^2 - A_{12}^2 - A_{12}A_{66}) + \frac{3}{4}\frac{A_{66}\pi^4}{a^2b^2\rho h} - \frac{2}{3}\frac{\pi^4 B_{11}}{b^4\rho h} - \frac{256b^2A_{66}A_{12}B_{11}}{3a^2m_1} + \frac{2a^2}{3b^2m_1}(A_{11}A_{66}^2 + 4A_{11}^2B_{11} - 4A_{12}^2B_{11}) + \frac{a^2A_{66}^2A_{11}}{12b^2m_2} + \frac{2a^4A_{11}A_{66}B_{11}}{3b^4m_1},$$

$$\varphi_{23} = \frac{4A_{66}}{3m_6}(2A_{11}^2 - 2A_{12}^2 - 3A_{12}A_{66}) + \frac{A_{66}}{12m_1}(65A_{11}^2 - 66A_{12}A_{66} + 128A_{11}B_{11} - 65A_{12}^2) + \frac{A_{66}}{6m_2}(A_{11}^2 - A_{12}^2 - A_{12}A_{66}) - \frac{1024b^4B_{11}A_{66}A_{11}}{3a^4m_1} + \frac{4a^4A_{11}A_{66}B_{11}}{3b^4m_1} + \frac{a^2A_{66}^2A_{11}}{6b^2m_2} + \frac{b^2}{3a^2m_1}(256A_{12}A_{66}B_{11} - 256A_{11}^2B_{11} + 128A_{66}^2B_{11} + A_{11}A_{66}^2 + 256A_{12}^2B_{11}) + \frac{64b^2A_{66}^2A_{11}}{3a^2m_6} + \frac{a^2}{48b^2m_1}(128A_{66}^2B_{11} - 256A_{12}^2B_{11} + 65A_{11}A_{66}^2 + 256A_{11}^2B_{11} - 128A_{12}A_{66}B_{11}) + \frac{a^2A_{66}^2A_{11}}{6b^2m_6} + \frac{A_{66}}{24a^2b^2A_{11}}(27A_{11} + 64B_{11}) - \frac{4}{3}\frac{\pi^4B_{11}}{b^4\rho h} + \frac{\pi^4}{6a^4\rho hA_{11}}(128B_{11}A_{11} + 17A_{12}A_{66}),$$

$$\varphi_{24} = \frac{1}{3m_1}(A_{12}^3 + 3A_{12}^2A_{66} - A_{11}^2A_{12} - A_{11}^2A_{66}) + \frac{1}{96}\frac{\pi^4}{a^2b^2\rho h}(7A_{12} - 60A_{66}) + \frac{a^4A_{11}^2A_{66}}{3b^4m_3} + \frac{1}{48m_3}(18A_{11}^2A_{66} - 48A_{12}A_{66}^2 - 4A_{12}^3 - 25A_{12}^2A_{66} + 4A_{11}^2A_{12}) - \frac{4b^2A_{66}A_{12}A_{11}}{3a^2m_1} + \frac{1}{96m_2}(23A_{11}^2A_{66} - 22A_{12}^2A_{66} - 96A_{12}A_{66}^2 + A_{12}^3 - A_{11}^2A_{12}) + \frac{1}{96}\frac{\pi^4}{b^4\rho hA_{11}}(13A_{11}^2 - 14A_{12}^2) - \frac{a^2A_{11}A_{12}A_{66}}{12b^2m_1} - \frac{a^4A_{11}^2A_{66}}{96b^4m_2} + \frac{b^2A_{66}A_{11}}{48a^2m_3}(6A_{66} + A_{12}) + \frac{b^2A_{66}A_{11}}{96a^2m_2}(48A_{66} - A_{12}) + \frac{a^2A_{11}}{12b^2m_3}(24A_{66}^2 + A_{11}^2 - A_{12}^2 + 2A_{12}A_{66}) + \frac{a^2A_{11}}{96b^2m_2}(48A_{66}^2 + A_{12}^2 + A_{12}A_{66} - A_{11}^2),$$

$$\varphi_{25} = \frac{1}{96m_4}(330A_{11}^2A_{66} - 432A_{12}A_{66}^2 - 433A_{12}^2A_{66} + 108A_{11}^2A_{12} - 108A_{12}^3) + \frac{27b^4A_{11}^2A_{66}}{16a^4m_4} + \frac{1}{48m_2}(43A_{11}^2A_{66} - 38A_{12}^2A_{66} - 16A_{12}A_{66}^2 - 5A_{12}^3 + 5A_{11}^2A_{12}) - \frac{32}{3}\frac{b^4A_{11}^2A_{66}}{a^4m_1} + \frac{1}{24m_5}(87A_{11}^2A_{66} + 36A_{11}^2A_{12} - 108A_{12}A_{66}^2 - 151A_{12}^2A_{66} - 36A_{12}^3) + \frac{b^4A_{11}^2A_{66}}{48a^4m_3}$$

$$
\begin{aligned}
&+\frac{1}{3m_1}(8A_{12}A_{66}^2+2A_{12}^2A_{66}+2A_{12}^3-2A_{11}^2A_{12})+\frac{7b^4A_{11}^2A_{66}}{24a^4m_2}+\frac{27b^4A_{11}^2A_{66}}{a^4m_5}\\
&+\frac{1}{96m_3}(106A_{11}^2A_{66}-48A_{12}A_{66}^2-12A_{12}^3-97A_{12}^2A_{66}+12A_{11}^2A_{12})+\frac{a^4A_{11}^2A_{66}}{6b^4m_4}\\
&+\frac{b^2}{32a^2m_4}(33A_{11}A_{12}A_{66}-24A_{11}A_{12}^2+162A_{11}A_{66}^2+24A_{11}^3)+\frac{a^2A_{11}A_{66}}{12b^2m_5}(2A_{12}+3A_{66})\\
&+\frac{b^2}{48a^2m_2}(8A_{11}A_{66}^2-14A_{11}A_{12}^2-23A_{11}A_{12}A_{66}+14A_{11}^3)-\frac{a^2A_{11}A_{66}}{6b^2m_1}(2A_{66}+A_{12})\\
&+\frac{b^2}{4a^2m_5}(12A_{11}^3-12A_{11}A_{12}^2+81A_{11}A_{66}^2+30A_{11}A_{12}A_{66})+\frac{a^4A_{11}^2A_{66}}{48b^4m_2}+\frac{a^4A_{11}^2A_{66}}{6b^4m_3}\\
&+\frac{8b^2A_{11}}{3a^2m_1}(A_{12}A_{66}+A_{12}^2-2A_{66}^2-A_{11}^2)+\frac{a^2A_{11}}{24b^2m_3}(A_{11}^2+24A_{66}^2-A_{12}^2+10A_{12}A_{66})\\
&+\frac{b^2}{96a^2m_3}(8A_{11}^3-8A_{11}A_{12}^2+6A_{11}A_{66}^2-13A_{11}A_{12}A_{66})+\frac{17}{24}\frac{\pi^4}{a^4\rho hA_{11}}(A_{11}^2-A_{12}^2)\\
&+\frac{a^2A_{11}}{8b^2m_4}(8A_{66}^2-3A_{12}^2+3A_{11}^2-2A_{12}A_{66})+\frac{a^2A_{11}}{48b^2m_2}(8A_{66}^2-A_{12}^2+A_{11}^2+3A_{12}A_{66})\\
&+\frac{1}{48}\frac{\pi^4}{b^4\rho hA_{11}}(19A_{11}^2-21A_{12}^2)-\frac{1}{24}\frac{\pi^4}{a^2b^2\rho h}(5A_{66}+4A_{12}),
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{26}=&\frac{A_{66}}{3m_1}(32A_{11}^2-24A_{12}A_{66}-32A_{12}^2)+\frac{1}{8}\frac{\pi^4}{b^4\rho hA_{11}}(4A_{11}^2-5A_{12}^2)-\frac{1}{12}\frac{\pi^4}{a^2b^2\rho h}(29A_{66}+2A_{12})\\
&+\frac{1}{24m_2}(12A_{11}^2A_{66}-17A_{12}^2A_{66}-14A_{12}A_{66}^2-4A_{12}^3+4A_{11}^2A_{12})+\frac{64b^2A_{66}^2A_{11}}{3a^2m_1}+\frac{a^4A_{11}^2A_{66}}{8b^4m_2}\\
&+\frac{b^2A_{66}A_{11}}{6a^2m_2}(2A_{66}+A_{12})+\frac{1}{24}\frac{a^2A_{11}}{b^2m_2}(6A_{66}^2-3A_{12}^2-2A_{12}A_{66}+3A_{11}^2)+\frac{2a^2A_{66}^2A_{11}}{3b^2m_1},
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{27}=&\frac{1}{3m_6}(8A_{11}^2A_{12}+16A_{11}^2A_{66}-8A_{12}^3-20A_{12}A_{66}^2-28A_{12}^2A_{66})-\frac{1}{24}\frac{\pi^4}{a^2b^2\rho h}(77A_{66}+9A_{12})\\
&+\frac{1}{12m_1}(69A_{11}^2A_{66}-70A_{12}A_{66}^2-71A_{12}^2A_{66}-2A_{12}^3+2A_{11}^2A_{12})+\frac{1024b^4A_{11}^2A_{66}}{3a^4m_6}\\
&+\frac{1}{12m_2}(4A_{11}^2A_{66}-7A_{12}^2A_{66}-6A_{12}A_{66}^2-2A_{12}^3+2A_{11}^2A_{12})+\frac{64}{3}\frac{b^2A_{11}}{a^2m_6}(3A_{66}^2+A_{11}^2-A_{12}^2)\\
&+\frac{1}{3}\frac{b^2A_{11}}{a^2m_1}(A_{11}^2-A_{12}^2+67A_{66}^2)+\frac{b^2A_{11}A_{66}}{6a^2m_2}(2A_{66}+A_{12})+\frac{a^2A_{11}A_{66}}{6b^2m_6}(A_{66}+A_{12})\\
&+\frac{1}{48}\frac{a^2A_{11}}{b^2m_1}(3A_{66}^2-A_{12}^2+A_{11}^2)+\frac{1}{12}\frac{a^2A_{11}}{b^2m_2}(2A_{66}^2-A_{12}^2+A_{11}^2)+\frac{a^4A_{11}^2A_{66}}{192b^4m_1}+\frac{a^4A_{11}^2A_{66}}{12b^4m_2}
\end{aligned}
$$

$$+\frac{17}{12}\frac{\pi^4}{a^4\rho h A_{11}}(A_{11}^2-2A_{12}^2-2A_{12}A_{66})+\frac{1}{192}\frac{\pi^4}{b^4\rho h A_{11}}(53A_{11}^2-70A_{12}^2)+\frac{4b^4A_{11}^2A_{66}}{3a^4m_1}, \quad (A.2)$$

where

$$m_1=\frac{\rho h}{\pi^4}(-8A_{12}a^2A_{66}b^2+4a^2A_{11}^2b^2+a^4A_{11}A_{66}-4A_{12}^2a^2b^2+16A_{66}b^4A_{11}),$$
$$m_2=\frac{\rho h}{\pi^4}(a^2A_{11}^2b^2+a^4A_{11}A_{66}+A_{66}b^4A_{11}-A_{12}^2a^2b^2-2A_{12}a^2A_{66}b^2),$$
$$m_3=\frac{\rho h}{\pi^4}(-4A_{12}^2a^2b^2+4a^2A_{11}^2b^2+16a^4A_{11}A_{66}-8A_{12}a^2A_{66}b^2+A_{66}b^4A_{11}),$$
$$m_4=\frac{\rho h}{\pi^4}(36a^2A_{11}^2b^2+16a^4A_{11}A_{66}+81A_{66}b^4A_{11}-36A_{12}^2a^2b^2-72A_{12}a^2A_{66}b),$$
$$m_5=\frac{\rho h}{\pi^4}(9a^2A_{11}^2b^2+a^4A_{11}A_{66}+81A_{66}b^4A_{11}-9A_{12}^2a^2b^2-18A_{12}a^2A_{66}b^2),$$
$$m_6=\frac{\rho h}{\pi^4}(16a^2A_{11}^2b^2+256A_{66}b^4A_{11}+a^4A_{11}A_{66}-16A_{12}^2a^2b^2-32A_{12}a^2A_{66}b). \quad (A.3)$$

Appendix B

The coefficients presented in Eq. (23) are as follows:

$$\eta=\frac{\zeta}{\rho h},\quad \alpha_{11}=\frac{\eta}{\tilde{\Omega}_{11}},\quad \alpha_{1i}=\frac{\tilde{\sigma}_{1(i-1)}}{\tilde{\Omega}_{11}}\frac{h^4\alpha}{a^2}\ (i=2,\ldots,5),\quad \beta_{11}=1,\quad \beta_{12}=2\tilde{\mu}_{11},\quad \beta_{13}=\frac{\tilde{\Omega}_{12}^2}{\tilde{\Omega}_{11}^2},$$
$$\beta_{14}=2\tilde{\mu}_{12}\frac{\tilde{\Omega}_{12}^2}{\tilde{\Omega}_{11}^2},\quad \gamma_{1i}=\frac{\tilde{\varphi}_{1i}}{\tilde{\Omega}_{11}^2}\frac{h^4\alpha}{a^2}\ (i=1,\ldots,7),\quad \alpha_{21}=\frac{\tilde{\sigma}_{21}}{\tilde{\Omega}_{11}}\frac{h^4\alpha}{a^2},\ \alpha_{22}=\frac{\tilde{\sigma}_{22}}{\tilde{\Omega}_{11}}\frac{h^4\alpha}{a^2},$$
$$\alpha_{23}=\frac{\eta}{\tilde{\Omega}_{11}},\quad \alpha_{24}=\frac{\tilde{\sigma}_{23}}{\tilde{\Omega}_{11}}\frac{h^4\alpha}{a^2},\quad \alpha_{25}=\frac{\tilde{\sigma}_{24}}{\tilde{\Omega}_{11}}\frac{h^4\alpha}{a^2},\quad \beta_{21}=\frac{\tilde{\Omega}_{21}^2}{\tilde{\Omega}_{11}^2},\quad \beta_{22}=2\tilde{\mu}_{21}\frac{\tilde{\Omega}_{21}^2}{\tilde{\Omega}_{11}^2},$$
$$\beta_{23}=\frac{\tilde{\Omega}_{22}^2}{\tilde{\Omega}_{11}^2},\quad \beta_{24}=2\tilde{\mu}_{22}\frac{\tilde{\Omega}_{22}^2}{\tilde{\Omega}_{11}^2},\quad \gamma_{2i}=\frac{\tilde{\varphi}_{2i}}{\tilde{\Omega}_{11}^2}\frac{h^4\alpha}{a^2}\ (i=1,\ldots,7), \quad (B.1)$$

where

$$\tilde{\sigma}_{11}=\frac{\pi^4k}{40a^3\rho h}\left(27+19\frac{A_{12}}{A_{11}}\alpha^2\right),\quad \tilde{\sigma}_{12}=-\frac{3\pi^4k}{20a^3\rho h}\left(10-\frac{A_{12}}{A_{11}}\alpha^2\right),\quad \tilde{\sigma}_{13}=\frac{3\pi^4k}{5a^3\rho h}\left(3+\frac{A_{12}}{A_{11}}\alpha^2\right),$$
$$\tilde{\sigma}_{14}=-\frac{3\pi^4k}{5a^3\rho h}\left(12+\frac{A_{12}}{A_{11}}\alpha^2\right),\quad \tilde{\sigma}_{21}=-\frac{3\pi^4k}{20a^3\rho h}\left(3+\frac{A_{12}}{A_{11}}\alpha^2\right),\quad \tilde{\sigma}_{22}=\frac{3\pi^4k}{20a^3\rho h}\left(12+\frac{A_{12}}{A_{11}}\alpha^2\right),$$
$$\tilde{\sigma}_{23}=-\frac{3\pi^4k}{5a^3\rho h}\left(2-3\frac{A_{12}}{A_{11}}\alpha^2\right),\quad \tilde{\sigma}_{24}=\frac{\pi^4k}{10a^3\rho h}\left(108-19\frac{A_{12}}{A_{11}}\alpha^2\right),$$

$$\tilde{\Omega}_{11}^2 = \frac{\pi^2}{a^4\rho h}\left[\frac{16}{15}\pi^2 D_{11}(9+5\alpha^4) + \frac{32}{5}\pi^2\alpha^2(D_{12}+2D_{66}) - \frac{4}{15}a^2\left(9+5\frac{A_{12}}{A_{11}}\alpha^2\right)N_0 - \frac{32}{15}\pi^2\frac{B_{11}^2}{A_{11}}(3+\alpha^4) + \frac{6k_0}{5k_1}\right],$$

$$\tilde{\Omega}_{12}^2 = \frac{\pi^2}{a^4\rho h}\left[\frac{32}{5}a^2N_0 - \frac{512}{5}\pi^2 D_{11} - \frac{256}{15}\pi^2\alpha^2(D_{12}+2D_{66}) + \frac{32B_{11}^2}{15A_{11}}\pi^2(32-\alpha^4) + 6\pi^2\alpha^2\frac{B_{11}A_{66}}{A_{11}} + \frac{4k_2}{5k_3}\right],$$

$$\tilde{\mu}_{11} = \frac{2\pi^2}{15a^2\rho h\tilde{\Omega}_{11}^2}\left(9+5\frac{A_{12}}{A_{11}}\alpha^2\right)N_1, \quad \tilde{\mu}_{12} = -\frac{16\pi^2}{5a^2\rho h\tilde{\Omega}_{12}^2}N_1,$$

$$\tilde{\Omega}_{21}^2 = \frac{\pi^2}{a^4\rho h}\left[\frac{8}{5}a^2N_0 - \frac{32}{5}\pi^2 D_{11} - \frac{64}{15}\pi^2\alpha^2(D_{12}+2D_{66}) + \frac{32B_{11}^2}{15A_{11}}\pi^2(2-\alpha^4) - \frac{4k_0}{5k_1}\right],$$

$$\tilde{\Omega}_{22}^2 = \frac{\pi^2}{a^4\rho h}\left[\frac{16}{15}\pi^2 D_{11}(144+5\alpha^4) - \frac{4}{15}\left(36+5\frac{A_{12}}{A_{11}}\alpha^2\right)a^2N_0 + \frac{128}{5}\pi^2\alpha^2(D_{12}+2D_{66}) - \frac{32B_{11}^2}{15A_{11}}\pi^2(48+\alpha^4) - \frac{16}{3}\pi^2\alpha^2\frac{B_{11}A_{66}}{A_{11}}\right],$$

$$\tilde{\mu}_{21} = -\frac{4\pi^2}{5a^2\rho h\tilde{\Omega}_{21}^2}N_1, \quad \tilde{\mu}_{22} = \frac{2\pi^2}{15a^2\rho h\tilde{\Omega}_{22}^2}\left(36+5\frac{A_{12}}{A_{11}}\alpha^2\right)N_1,$$

$$\tilde{\varphi}_{1i} = \frac{6}{5}\varphi_{1i} - \frac{4}{5}\varphi_{2i}\ (i=1,\ldots,7), \quad \tilde{\varphi}_{2i} = \frac{6}{5}\varphi_{2i} - \frac{4}{5}\varphi_{1i}\ (i=1,\ldots,7) \tag{B.2}$$

and

$$k_0 = -\frac{8}{3}B_{11}^2\pi^2\left[A_{66}\left(\alpha^6+\frac{1}{\alpha^2}\right) + A_{11}(1+\alpha^4) + 2(A_{12}+A_{66})\alpha^2\right],$$

$$k_1 = (A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) + A_{11}A_{66}\left(\alpha^2+\frac{1}{\alpha^2}\right),$$

$$k_2 = A_{66}B_{11}^2\left(\alpha^6+\frac{256}{\alpha^2}\right) + B_{11}(A_{66}^2+4A_{11}B_{11})\alpha^4 + 16B_{11}(A_{12}A_{66}+A_{66}^2+4A_{11}B_{11}) + 4B_{11}(A_{11}A_{66}+8A_{66}B_{11}+8A_{12}B_{11})\alpha^2,$$

$$k_3 = 4(A_{11}^2 - A_{12}^2 - 2A_{12}A_{66}) + A_{11}A_{66}\left(\alpha^2+\frac{16}{\alpha^2}\right). \tag{B.3}$$

Appendix C

The coefficients in Eqs. (39) and (44) are given as

$$
\begin{aligned}
&a_1 = -\frac{\alpha_{11}}{2}, \quad \tilde{a}_1 = -\sigma_1, \quad b_1 = -\frac{\alpha_{12}}{2}, \quad \tilde{b}_1 = \frac{1}{2\omega}\left[-3\gamma_{11} + \frac{10\gamma_{15}^2}{3\omega^2} + \frac{\gamma_{16}(6\gamma_{15}-\gamma_{25})}{3\omega^2}\right],\\
&\tilde{c}_1 = \frac{1}{\omega}\left[-\gamma_{12} + \frac{2\gamma_{15}\gamma_{16}}{3\omega^2} + \frac{\gamma_{16}(3\gamma_{15}+\gamma_{26})}{3\omega^2} + \frac{2\gamma_{17}\gamma_{25}}{\omega^2}\right],\\
&\tilde{d}_1 = \frac{1}{\omega}\left[-\gamma_{13} + \frac{2\gamma_{15}\gamma_{17}}{\omega^2} + \frac{\gamma_{16}(\gamma_{16}+3\gamma_{27})}{3\omega^2} + \frac{2\gamma_{17}\gamma_{26}}{3\omega^2}\right], \quad e_1 = -\frac{\alpha_{15}}{2},\\
&\tilde{e}_1 = \frac{1}{2\omega}\left(-3\gamma_{14} + \frac{5\gamma_{16}\gamma_{17}}{3\omega^2} + \frac{10\gamma_{17}\gamma_{27}}{3\omega^2}\right), \quad \tilde{f}_1 = -\frac{1}{2\omega}\left[\gamma_{12} - \frac{2\gamma_{15}\gamma_{16}}{\omega^2} + \frac{\gamma_{16}(\gamma_{15}-3\gamma_{26})}{3\omega^2} + \frac{2\gamma_{17}\gamma_{25}}{3\omega^2}\right],\\
&f_1 = -\frac{\alpha_{13}}{2}, \quad g_1 = -\frac{\alpha_{14}}{2}, \quad \tilde{g}_1 = -\frac{1}{2\omega}\left[\gamma_{13} + \frac{2\gamma_{15}\gamma_{17}}{3\omega^2} - \frac{\gamma_{16}(3\gamma_{16}-\gamma_{27})}{3\omega^2} - \frac{2\gamma_{17}\gamma_{26}}{\omega^2}\right], \quad \tilde{h}_1 = -\frac{\beta_{13}}{2\omega},\\
&\tilde{i}_1 = \frac{\beta_{12}}{4\omega}, \quad \tilde{j}_1 = \frac{\beta_{14}}{4\omega}, \quad a_2 = -\frac{\alpha_{23}}{2}, \quad \tilde{a}_2 = -\sigma_2, \quad \tilde{b}_2 = \frac{1}{2\omega}\left[-3\gamma_{21} + \frac{10\gamma_{15}\gamma_{25}}{3\omega^2} + \frac{\gamma_{26}(6\gamma_{15}-\gamma_{25})}{3\omega^2}\right],\\
&b_2 = -\frac{\alpha_{21}}{2}, \quad \tilde{c}_2 = \frac{1}{\omega}\left[-\gamma_{22} + \frac{2\gamma_{16}\gamma_{25}}{3\omega^2} + \frac{\gamma_{26}(3\gamma_{15}+\gamma_{26})}{3\omega^2} + \frac{2\gamma_{25}\gamma_{27}}{\omega^2}\right],\\
&\tilde{d}_2 = \frac{1}{\omega}\left[-\gamma_{23} + \frac{2\gamma_{17}\gamma_{25}}{\omega^2} + \frac{\gamma_{26}(\gamma_{16}+3\gamma_{27})}{3\omega^2} + \frac{2\gamma_{26}\gamma_{27}}{3\omega^2}\right], \quad e_2 = -\frac{\alpha_{25}}{2},\\
&\tilde{e}_2 = -\frac{1}{2\omega}\left(3\gamma_{24} - \frac{5\gamma_{17}\gamma_{26}}{3\omega^2} - \frac{10\gamma_{27}^2}{3\omega^2}\right), \quad f_2 = -\frac{\alpha_{22}}{2},\\
&\tilde{f}_2 = -\frac{1}{2\omega}\left[\gamma_{22} - \frac{2\gamma_{16}\gamma_{25}}{\omega^2} + \frac{\gamma_{26}(\gamma_{15}-3\gamma_{26})}{3\omega^2} + \frac{2\gamma_{25}\gamma_{27}}{3\omega^2}\right], \quad g_2 = -\frac{\alpha_{24}}{2},\\
&\tilde{g}_2 = -\frac{1}{2\omega}\left[\gamma_{23} + \frac{2\gamma_{17}\gamma_{25}}{3\omega^2} - \frac{\gamma_{26}(3\gamma_{16}-\gamma_{27})}{3\omega^2} - \frac{2\gamma_{26}\gamma_{27}}{\omega^2}\right],\\
&\tilde{h}_2 = -\frac{\beta_{21}}{2\omega}, \quad \tilde{i}_2 = \frac{\beta_{22}}{4\omega}, \quad \tilde{j}_2 = \frac{\beta_{24}}{4\omega}
\end{aligned} \tag{C.1}
$$

Appendix D

The coefficients $C_{ij}(i = 1, \ldots, 4; j = 1, \ldots, 4)$ of Eqs. (49) are given as follows:

$$
\begin{aligned}
C_{11} = {} & 2g_1\chi_0^2\cos(\alpha_0-\beta_0)^2 + \tilde{i}_1\sin(2\alpha_0) + \tilde{g}_1\chi_0^2\sin(2\alpha_0-2\beta_0) + 2f_1\rho_0\chi_0\cos(\alpha_0-\beta_0)\\
& - 2\tilde{f}_1\rho_0\chi_0\sin(\alpha_0-\beta_0) - g_1\chi_0^2 + 2\tilde{c}_1\rho_0\chi_0\sin(\alpha_0-\beta_0) + 3b_1\rho_0^2 + a_1,
\end{aligned}
$$

$$
\begin{aligned}
C_{12} = {} & 3e_1\chi_0^2\cos(\alpha_0-\beta_0) + 3\tilde{e}_1\chi_0^2\sin(\alpha_0-\beta_0) + f_1\rho_0^2\cos(\alpha_0-\beta_0) - 2g_1\rho_0\chi_0\\
& - \tilde{f}_1\rho_0^2\sin(\alpha_0-\beta_0) + \tilde{c}_1\rho_0^2\sin(\alpha_0-\beta_0) + \tilde{h}_1\sin(\alpha_0-\beta_0) + \tilde{j}_1\sin(\alpha_0+\beta_0)\\
& + 2\tilde{g}_1\rho_0\chi_0\sin(2\alpha_0-2\beta_0) + 4g_1\rho_0\chi_0\cos(\alpha_0-\beta_0)^2,
\end{aligned}
$$

$$
\begin{aligned}
C_{13} = & -e_1\chi_0^3\sin(\alpha_0-\beta_0)+\tilde{e}_1\chi_0^3\cos(\alpha_0-\beta_0)+2\tilde{i}_1\rho_0\cos(2\alpha_0)+\tilde{j}_1\chi_0\cos(\alpha_0+\beta_0)\\
& +\tilde{h}_1\chi_0\cos(\alpha_0-\beta_0)-2g_1\rho_0\chi_0^2\sin(2\alpha_0-2\beta_0)-\tilde{f}_1\rho_0^2\chi_0\cos(\alpha_0-\beta_0)\\
& -f_1\rho_0^2\chi_0\sin(\alpha_0-\beta_0)+\tilde{c}_1\rho_0^2\chi_0\cos(\alpha_0-\beta_0)+2\tilde{g}_1\rho\chi_0^2\cos(2\alpha_0-2\beta_0),
\end{aligned}
$$

$$
\begin{aligned}
C_{14} = & \, e_1\chi_0^3\sin(\alpha_0-\beta_0)-\tilde{e}_1\chi_0^3\cos(\alpha_0-\beta_0)+\tilde{j}_1\chi_0\cos(\alpha_0+\beta_0)-\tilde{h}_1\chi_0\cos(\alpha_0-\beta_0)\\
& +2g_1\rho_0\chi_0^2\sin(2\alpha_0-2\beta_0)+\tilde{f}_1\rho_0^2\chi_0\cos(\alpha_0-\beta_0)+f_1\rho_0^2\chi_0\sin(\alpha_0-\beta_0)\\
& -\tilde{c}_1\rho_0^2\chi_0\cos(\alpha_0-\beta_0)-2\tilde{g}_1\rho_0\chi_0^2\cos(2\alpha_0-2\beta_0),
\end{aligned}
$$

$$
\begin{aligned}
C_{21} = & -2\tilde{f}_2\rho_0\chi_0\sin(2\alpha_0-2\beta_0)+\tilde{i}_2\sin(\alpha_0-\beta_0)-\tilde{h}_2\sin(\alpha_0-\beta_0)-\tilde{d}_2\chi_0^2\sin(\alpha_0-\beta_0)\\
& -3\tilde{b}_2\rho_0^2\sin(\alpha_0-\beta_0)+3b_2\rho_0^2\cos(\alpha_0-\beta_0)+\tilde{g}_2\chi_0^2\sin(\alpha_0-\beta_0)\\
& +g_2\chi_0^2\cos(\alpha_0-\beta_0)+2f_2\rho_0\chi_0\cos(2\alpha_0-2\beta_0),
\end{aligned}
$$

$$
\begin{aligned}
C_{22} = & -f_2\rho_0^2+3e_2\chi_0^2-\tilde{f}_2\rho_0^2\sin(2\alpha_0-2\beta_0)+2\tilde{g}_2\rho_0\chi_0\sin(\alpha_0-\beta_0)+2g_2\rho_0\chi_0\cos(\alpha_0-\beta_0)\\
& -2\tilde{d}_2\rho_0\chi_0\sin(\alpha_0-\beta_0)+2f_2\rho_0^2\cos(\alpha_0-\beta_0)^2+\tilde{j}_2\sin(2\beta_0)+a_2,
\end{aligned}
$$

$$
\begin{aligned}
C_{23} = & -\tilde{b}_2\rho_0^3\cos(\alpha_0-\beta_0)-b_2\rho_0^3\sin(\alpha_0-\beta_0)-\tilde{h}_2\rho_0\cos(\alpha_0-\beta_0)+\tilde{i}_2\rho_0\cos(\alpha_0-\beta_0)\\
& +\tilde{g}_2\rho_0\chi_0^2\cos(\alpha_0-\beta_0)-2f_2\rho_0^2\chi_0\sin(2\alpha_0-2\beta_0)-2\tilde{f}_2\rho_0^2\chi_0\cos(2\alpha_0-2\beta_0)\\
& -\tilde{d}_2\rho_0\chi_0^2\cos(\alpha_0-\beta_0)-g_2\rho_0\chi_0^2\sin(\alpha_0-\beta_0),
\end{aligned}
$$

$$
\begin{aligned}
C_{24} = & \, \tilde{b}_2\rho_0^3\cos(\alpha_0-\beta_0)+b_2\rho_0^3\sin(\alpha_0-\beta_0)+\tilde{h}_2\rho_0\cos(\alpha_0-\beta_0)-\tilde{i}_2\rho_0\cos(\alpha_0-\beta_0)\\
& +2\tilde{j}_2\chi_0\cos(2\beta_0)-\tilde{g}_2\rho_0\chi_0^2\cos(\alpha_0-\beta_0)+2f_2\rho_0^2\chi_0\sin(2\alpha_0-2\beta_0)\\
& +2\tilde{f}_2\rho_0^2\chi_0\cos(2\alpha_0-2\beta_0)+\tilde{d}_2\rho_0\chi_0^2\cos(\alpha_0-\beta_0)+g_2\rho_0\chi_0^2\sin(\alpha_0-\beta_0),
\end{aligned}
$$

$$
\begin{aligned}
C_{31} = & \, 2\tilde{b}_1\rho_0+(\tilde{c}_1+\tilde{f}_1)\chi_0\cos(\alpha_0-\beta_0)-\frac{\tilde{j}_1\chi_0}{\rho_0^2}\cos(\alpha_0-\beta_0)+\frac{e_1\chi_0^3}{\rho_0^2}\sin(\alpha_0-\beta_0)\\
& -\frac{(\tilde{e}_1+\tilde{h}_1)\chi_0^3}{\rho_0^2}\cos(\alpha_0-\beta_0)+f_1\chi_0\sin(\alpha_0-\beta_0),
\end{aligned}
$$

$$
\begin{aligned}
C_{32} = & -2g_1\chi_0\sin(2\alpha_0-2\beta_0)+2\tilde{g}_1\chi_0\cos(2\alpha_0-2\beta_0)+f_1\rho_0\sin(\alpha_0-\beta_0)\\
& (\tilde{f}_1+\tilde{c}_1)\rho_0\cos(\alpha_0-\beta_0)+\frac{(\tilde{h}_1+\tilde{j}_1+3\tilde{e}_1\chi_0^2)}{\rho_0}\cos(\alpha_0-\beta_0)-\frac{3e_1\chi_0^2}{\rho_0}\sin(\alpha_0-\beta_0)+2\tilde{d}_1\chi_0,
\end{aligned}
$$

$$
\begin{aligned}
C_{33} = & -2\tilde{i}_1\sin(2\alpha_0)-(\tilde{f}_1+\tilde{c}_1)\rho_0\chi_0\sin(\alpha_0-\beta_0)-\frac{\tilde{e}_1\chi_0^3}{\rho_0}\sin(\alpha_0-\beta_0)-2g_1\chi_0^2\cos(2\alpha_0-2\beta_0)\\
& -2\tilde{g}_1\chi_0^2\sin(2\alpha_0-2\beta_0)-\tilde{h}_1\chi_0/\rho_0\sin(\alpha_0-\beta_0),
\end{aligned}
$$

$$C_{34} = (\tilde{f}_1 + \tilde{c}_1)\rho_0\chi_0 \sin(\alpha_0 - \beta_0) - \tilde{j}_1\chi_0/\rho_0 \sin(\alpha_0 + \beta_0) + \tilde{e}_1\chi_0^3/\rho_0 \sin(\alpha_0 - \beta_0) + 2g_1\chi_0^2 \cos(2\alpha_0 - 2\beta_0) + \tilde{h}_1\chi_0/\rho_0 \sin(\alpha_0 - \beta_0) + e_1\chi_0^3/\rho_0 \cos(\alpha_0 - \beta_0) - f_1\rho_0\chi_0 \cos(\alpha_0 - \beta_0) + 2\tilde{g}_1\chi_0^2 \sin(2\alpha_0 - 2\beta_0),$$

$$C_{41} = -g_2\chi_0 \sin(\alpha_0 - \beta_0) + \tilde{d}_2\chi_0 \cos(\alpha_0 - \beta_0) + 3\tilde{b}_2\rho_0^2/\chi_0 \cos(\alpha_0 - \beta_0) + 3b_2\rho_0^2/\chi_0 \sin(\alpha_0 - \beta_0) 2f_2\rho_0 \sin(2\alpha_0 - 2\beta_0) + 2\tilde{f}_2\rho_0 \cos(2\alpha_0 - 2\beta_0) + \tilde{g}_2\chi_0 \cos(\alpha_0 - \beta_0) + \tilde{h}_2/\chi_0 \cos(\alpha_0 - \beta_0) + \tilde{i}_2/\chi_0 \cos(\alpha_0 + \beta_0) + 2\tilde{c}_2\rho_0,$$

$$C_{42} = 2\tilde{e}_2\chi_0 + (\tilde{d}_2 + \tilde{g}_2)\rho_0 \cos(\alpha_0 - \beta_0) - g_2\rho_0 \sin(\alpha_0 - \beta_0) + \frac{\tilde{b}_2\rho_0^3}{\chi_0^2} \cos(\alpha_0 - \beta_0) - \frac{\tilde{i}_2\rho_0}{\chi_0^2} \cos(\alpha_0 + \beta_0) - \frac{b_2\rho_0^3}{\chi_0^2} \sin(\alpha_0 - \beta_0) - \frac{\tilde{h}_2\rho_0}{\chi_0^2} \cos(\alpha_0 - \beta_0),$$

$$C_{43} = -2\tilde{f}_2\rho_0^2 \sin(2\alpha_0 - 2\beta_0) - (\tilde{g}_2 + \tilde{d}_2)\rho_0\chi_0 \sin(\alpha_0 - \beta_0) - g_2\rho_0\chi_0 \cos(\alpha_0 - \beta_0) + 2f_2\rho_0^2 \cos(2\alpha_0 - 2\beta_0) - \frac{\tilde{b}_2\rho_0^3}{\chi_0} \sin(\alpha_0 - \beta_0) - \frac{\tilde{i}_2\rho_0}{\chi_0} \sin(\alpha_0 + \beta_0) + \frac{b_2\rho_0^3}{\chi_0} \cos(\alpha_0 - \beta_0) - \frac{\tilde{h}_2\rho_0}{\chi_0} \sin(\alpha_0 - \beta_0),$$

$$C_{44} = 2\tilde{f}_2\rho_0^2 \sin(2\alpha_0 - 2\beta_0) + (\tilde{g}_2 + \tilde{d}_2)\rho_0\chi_0 \sin(\alpha_0 - \beta_0) + g_2\rho_0\chi_0 \cos(\alpha_0 - \beta_0) - 2f_2\rho_0^2 \cos(2\alpha_0 - 2\beta) - 2\tilde{j}_2 \sin(2\beta_0) + \frac{\tilde{b}_2\rho_0^3}{\chi_0} \sin(\alpha_0 - \beta_0) - \frac{\tilde{i}_2\rho_0}{\chi_0} \sin(\alpha_0 + \beta_0) - \frac{b_2\rho_0^3}{\chi_0} \cos(\alpha_0 - \beta_0) + \frac{\tilde{h}_2\rho_0}{\chi_0} \sin(\alpha_0 - \beta_0). \quad \text{(D.1)}$$

References

[1] M. Ye, J. Lu, Q. Ding, W. Zhang, Nonlinear dynamic behavior of a parametrically excited rectangular symmetric cross-ply laminated composite plate, *Acta Mechanica Sinica* 37 (2004) 64–71.

[2] M. Ye, J. Lu, Q. Ding, W. Zhang, Local and global nonlinear dynamics of a parametrically excited rectangular symmetric cross-ply laminated composite plate, submitted for publication.

[3] A. Bhimaraddi, Large amplitude vibrations of imperfect antisymmetric angle-ply laminated plates, *Journal of Sound and Vibration* 162 (1993) 457–470.

[4] Y.S. Shih, P.T. Blotter, Non-linear vibration analysis of arbitrarily laminated rectangular thin plates on elastic foundations, *Journal of Sound and Vibration* 167 (1993) 433–459.

[5] N.K. Chandiramani, L.I. Librescu, R.H. Plaut, Flutter of geometrically imperfect shear-deformable laminated flat panels using non-linear aerodynamics, *Journal of Sound and Vibration* 192 (1996) 79–100.

[6] K. Oh, A.H. Nayfeh, Nonlinear resonances in cantilever composite plates, *Nonlinear Dynamics* 11 (1996) 143–169.

[7] A. Abe, Y. Kobayashi, G. Yamada, Two-mode response of simply supported, rectangular laminated plates, *International Journal of Non-linear Mechanics* 33 (1998) 675–690.

[8] P. Ribeiro, M. Petyt, Multi-modal geometrical non-linear free vibration of fully clamped composite laminated plates, *Journal of Sound and Vibration* 225 (1999) 127–152.
[9] M. Ganapathi, P. Boisse, D. Solaut, Non-linear dynamic stability analysis of composite laminated under periodic in-plane compressive loads, *International Journal for Numerical Methods in Engineering* 46 (1999) 943–956.
[10] M. Ganapathi, B.P. Patel, P. Boisse, M. Touratier, Non-linear dynamic stability characteristics of elastic plates subjected periodic in-plane load, *International Journal of Non-Linear Mechanics* 35 (2000) 467–480.
[11] B. Harras, R. Benamar, Geometrically non-linear free vibration of fully clamped symmetrically laminated rectangular composite plates, *Journal of Sound and Vibration* 251 (2002) 579–619.
[12] H. Tanriover, E. Senocak, Large deflection analysis of unsymmetrically laminated composite plates: analytical–numerical type approach, *International Journal of Non-Linear Mechanics* 39 (2004) 1385–1392.
[13] J. Hadian, A.H. Nayfeh, Modal interaction in circular plates, *Journal of Sound and Vibration* 142 (1990) 279–292.
[14] Z.C. Feng, P.R. Sethna, Global bifurcations in the motion of parametrically excited thin plate, *Nonlinear Dynamics* 4 (1993) 389–408.
[15] S.I. Chang, A.K. Bajaj, C.M. Krousgrill, Non-linear vibrations and chaos in harmonically excited rectangular plates with one-to-one internal resonance, *Nonlinear Dynamics* 4 (1993) 433–460.
[16] S. Sassi, G.L. Ostiguy, Effects of initial geometric imperfections on the interaction between forced and parametric vibrations, *Journal of Sound and Vibration* 178 (1994) 41–54.
[17] N. Malhotra, N. Sri Namachchivaya, Chaotic dynamics of shallow arch structures under 1:1 internal resonance, *ASCE Journal of Engineering Mechanics* 123 (1997) 620–627.
[18] A.A. Popov, J.M. Thompson, J.G. Croll, Bifurcation analyses in the parametrically excited vibrations of cylindrical panels, *Nonlinear Dynamics* 17 (1998) 205–225.
[19] G. Anlas, O. Elbeyli, Nonlinear vibrations of a simply supported rectangular metallic plate subjected to transverse harmonic excitation in presence of a one-to-one internal resonance, *Nonlinear Dynamics* 30 (1998) 1–28.
[20] W. Zhang, Global and chaotic dynamics for a parametrically excited thin plate, *Journal of Sound and Vibration* 239 (2001) 1013–1036.
[21] N.N. Bogoliubov, Y.A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon and Breach, New York, 1961.
[22] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*, Wiley-Interscience, New York, 1979.
[23] A.H. Nayfeh, B. Balachandran, *Applied Nonlinear Dynamics*, Wiley-Interscience, New York, 1994.
[24] A. Maccari, Approximate solution of a class of nonlinear oscillators in resonance with a periodic excitation, *Nonlinear Dynamics* 15 (1998) 329–343.
[25] A. Maccari, Dissipative bidimensional systems and resonant excitation, *International Journal of Non-Linear Mechanics* 33 (1998) 713–726.
[26] A. Maccari, The asymptotic perturbation method for nonlinear continuous systems, *Nonlinear Dynamics* 19 (1999) 1–18.
[27] A. Maccari, Modulated motion and infinite-period homoclinic bifurcation for parametrically excited Lienard systems, *International Journal of Non-Linear Mechanics* 35 (2000) 239–262.
[28] A. Maccari, Multiple resonant or non-resonant parametric excitations for nonlinear oscillators, *Journal of Sound and Vibration* 37 (2001) 855–866.
[29] G.T. Lim, J.N. Reddy, On canonical bending relationships for plates, *International Journal of Solids and Structures* 40 (2003) 3039–3067.
[30] C.Y. Chia, *Non-Linear analysis of Plate*, McGraw-Hill, New York, 1980.
[31] J.N. Reddy, *Mechanics of Laminated Composite Plates, Theory and Analysis*, CRC Press, New York, 1997.
[32] R.M. Jones, *Mechanics of Composite Materials*, second ed., McGraw-Hill, New York, 1999.
[33] M. Ye, L.X. Xiao, *Analytical Mechanics*, Tianjin University Press, Tianjin, 2001.
[34] H.E. Nusse, J.A. Yorke, *Dynamics: Numerical Explorations*, Springer, New York, Berlin, 1997.